Data, Logic, and Computing

ECS 17 (Winter 2024)

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Final: solutions

Part 1: Data *(10 questions, each 3 points; total 30 points)*

1) *1) Which binary number is* $(101111)_2 + 1$ *equal to?*

   a) $(101112)_2$
   b) $(101110)_2$
   c) $(111111)_2$
   d) $(110000)_2$

   $(101111)_2 = (47)_{10}$, therefore $(101111)_2 + 1 = (48)_{10} = (110000)_2$.

2) *2) How much space would you need to store a 10 min song that has been sampled at 44.1 KHz, with each data point stored on 8 bits, in quadrophony (i.e. using 4 microphones to record the song; assume no compression)?*

   a) *About 100 MBytes*
   b) *About 100 Mbits*
   c) *About 50 MBytes*
   d) *About 5 MBytes*

   Storage: 10 (mins) $\times$ 60 (second/min) $\times$ 44100 (samples) $\times$ 8 (bits) $\times$ 4 (quadrophony) = 846720000 (bits) = 105840000 bytes $\approx$ 100 megabytes.

3) $(1101)_2 - (110)_2 =$

   a) *#B*
   b) *#7*
   c) *#A*
   d) *#6*

   Note that:

a) $(1101)_2 = (13)_{10}$

b) $(110)_2 = (6)_{10}$

Therefore $(1101)_2 - (110)_2 = (7)_{10} = \#7$.

4) *$\#C9 - \#B3 =$*

a) $(10110)_2$

b) $(11110)_2$

c) $(10111)_2$

d) $(11111)_2$

Note that:

a) $\#C9 = (201)_{10}$

b) $\#B3 = (179)_{10}$

Therefore $\#C9 - \#B3 = (22)_{10} = (10110)_2$.

5) *Assuming that there are 400,000 characters in the UNICODE, what is the minimal number of bits needed to store a word of 8 characters with this code?*

a) *19*

b) *190*

c) *152*

d) *148*

There are 400,000 characters: to represent any of the characters, we need at least 19 bits ($2^{18} = 262144$, $2^{19} = 524288$). As the word contains 8 characters, we need at least $8 \times 19 = 152$ bits.

6) *If a signal is thought to have a maximum frequency between 1000 Hz and 4000 Hz, which of the following would be the most appropriate sample rate?*

a) *500 Hz*

b) *4000 Hz*

c) *8000 Hz*

d) *9000 Hz*

The largest frequency in the signal is 4000 Hz; the sampling rate needs to be strictly greater than twice this frequency, is $f_s > 8000$ Hz.

7) *As you are reading a paper, you see the word LOA_ (upper case), where the underscore is unfortunately a letter you cannot read. You know, however, that the sum of all ASCII identifiers for this word is #120. What is the missing letter represented as the underscore*

a) *S*

b) *F*

c) *D*

d) *N*

The letters L, O, and A are encoded with the decimal numbers 76, 79, and 65, respectively. The sum of the ascii identifier is #120 = 288. Therefore the missing letter is represented with the decimal number $288 - 76 - 79 - 65 = 68$, which is the letter D.

8) *Let A be the hexadecimal number #$F1$ and B the hexadecimal number #101; which of these hexadecimal numbers C satisfies $A+C = B$*

   a) #$A0$

   b) #10

   c) #11

   d) #$1F2$

Note that:

a) $\#F1 = (241)_{10}$

b) $\#101 = (257)_{10}$

Therefore $A = 257 - 241 = (16)_{10} = \#10$.

9) *Mannikins are birds that can flap their wings up to 100 times a second. Which of these sampling rates is most appropriate to use if you want to monitor the flight of a mannikin correctly with a digital device?*

   a) *3 Hz*

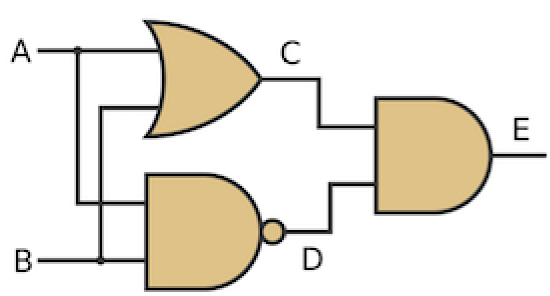
   b) *30 Hz*

   c) *300 Hz*

   d) *100 Hz*

The maximum flapping rate for a mannikin is 100 beats per second, is 100 Hz. The sampling rate needs to be strictly larger than twice this rate, i.e. $f_s > 200$ Hz.

10) *The gate shown below is equivalent to:*



   a) *The AND gate*

b) *The OR gate*
c) *The XOR gate*
d) *the XNOR gate*

This is the XOR gate.

Part II *3 problems, each 10 points; total 30 points)*

1) *For each of the five propositions in the table below, indicates on the right if they are always tautologies or not (x is a real number, p and q are propositions, T is a tautology, and F is a contradiction).* (10 points)

Table 1: Propositional logic

Proposition	Tautology (Yes/ No)
If $x^4 = -x^2 - 1$, then $25 = 23 + 3$	Yes: This is $p \to q$ where p is false (x^4 is always positive, while $-x^2 - 1$ is always strictly negative: therefore $p \to q$ is true
$(p \wedge \neg p) \to q$	Yes: $p \wedge \neg p$ is always false, therefore the implication is true.
$(p \vee F) \to p$	Yes: $(p \vee F)$ is equivalent to p, and $p \to p$ is always true.
$(p \to q) \leftrightarrow (\neg p \vee q)$	Yes! Note that $p \to q$ and $\neg p \vee q$ are logically equivalent.
$(p \wedge \neg p) \vee (\neg p \wedge q)$	No (use table)

2) *Inspector Craig from Scotland Yard has been assigned a special mission: identify a knave on an island that otherwise is inhabited only by knights. This island, however, is unusual: knights always tell the truth in the morning and always lie in the afternoon, while knaves always lie in the morning and tell the truth in the afternoon. As he arrives on the island, he is presented with 3 inhabitants, Alice, Ben, and Claire. He is told that one of them is the knave he is looking for (the two others are then knights). Unfortunately, he does not know if it is currently morning or afternoon. Alice tells him, "if I am a knave, it is currently morning", while Claire tells him that "Alice is a knave". Can you help Inspector Craig find the knave? Can you also tell him if it is morning or afternoon? (10 points)*

Alice, Ben, and Claire can be knights or knaves, but one of them is a knave, and the two others are knights. It can be morning or afternoon. There are therefor 6 cases to consider

Line number	Alice	Ben	Claire	Time of day	Alice says	Claire says
1	Knight	Knight	Knave	Morning	T	F
2	Knight	Knight	Knave	Afternoon	T	F
3	Knight	Knave	Knight	Morning	T	F
4	Knight	Knave	Knight	Afternoon	T	F
5	Knave	Knight	Knight	Morning	T	T
6	Knave	Knight	Knight	Afternoon	F	T

We can eliminate:

- Line 2, as Alice would be a knight that tells the truth in the afternoon
- Line 3, as Claire would be a knight that lies in the morning
- Line 4, as Alice would be a knight that tells the truth in the afternoon
- Line 5, as Alice would be a knave that tells the truth in the morning
- Line 6, as Alice would be a knave that lies in the afternoon

The only option is Line 1; Claire is the knave, and it is currently morning.

3) *Let p and q be two propositions. Use a truth table or logical equivalence to indicate if the proposition $(p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$ is a tautology, a contradiction, or neither*

Let us build the Truth table for $A = (p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$

p	q	$\neg p$	$\neg q$	$p \wedge q$	$p \wedge \neg q$	$\neg p \wedge q$	$\neg p \wedge \neg q$	$(p \wedge q) \vee (p \wedge \neg q)$	$(\neg p \wedge q) \vee (\neg p \wedge \neg q)$	A
T	T	F	F	T	F	F	F	T	F	T
T	F	F	T	F	T	F	F	T	F	T
F	T	T	F	F	F	T	F	F	T	T
F	F	T	T	F	F	F	T	F	T	T

From Column 11, we can conclude that A is a tautology.

Part III *4 problems, each 10 points; total 40 points)*

1) *Let n be an integer. Show that n is odd if and only if $n^2 - 1$ is a multiple of 8 (note that n is a multiple of 8 if and only if there exists an integer k such that $n = 8k$).*

Let p be the proposition "n is odd" and q be the proposition "$n^2 - 1$ is a multiple of 8". We want to show that $p \leftrightarrow q$ is true.

i) Let us show $p \to q$ using a direct proof:
Hypothesis: p is true, i.e. n is odd. There exists an integer k such that let $n = 2k+1$,We get:

$$\begin{aligned} n^2 - 1 &= (2k+1)^2 - 1 \\ &= 4k^2 + 4k \\ &= 4k(k+1) \end{aligned}$$

Note that from class, we know that $k(k+1)$ is always even, as k is an integer. Therefore, there exists an integer l such that $k(k+1) = 2l$. Then,

$$n^2 - 1 = 8l$$

hence $n^2 - 1$ is a multiple of 8 as l is an integer. We have shown that q is true, therefore $p \to q$ is true.

ii) Let us show $q \rightarrow p$ using a direct proof:
We assume that $\neg q$ is true, i.e. that $n^2 - 1$ is a multiple of 8. Therefore, there exists an integer k such that $n^2 - 1 = 8k$. Therefore,

$$\begin{aligned} n^2 &= 8k + 1 \\ &= 2(4k) + 1 \end{aligned}$$

As $4k$ is an integer, n^2 is odd. From class, we know then that n is odd. We have shown that p is true, therefore $q \rightarrow p$ is true.

2) *Use induction to prove that any postage value of n cents can be made with only 5-cent stamps and 7-cent stamps, whenever $n \geq 24$, n natural number.*

Let $p(n)$ be the proposition that n cents can be made with only 5–cent and 7–cent stamps, when n is greater than or equal to 24.
This means that there exists two positive integers a_n and b_n such that $n = 5a_n + 7b_n$

a) Basis step: I want to prove that $p(24)$ is true
Note that: $24 = 2 \times 5 + 2 \times 7$
We can set $a_{24} = 2$ and $b_{24} = 2$. Both are positive integers, and $24 = 5a_{24} + 7b_{24}$. Therefore $p(24)$ is true.

b) Inductive Step
I want to show $p(n) \rightarrow p(n+1)$ whenever $n \geq 24$
Hypothesis: $p(n)$ is true and there exists two positive integers a_n and b_n such that $n = 5a_n + 7b_n$
Then:
$n + 1 = 5a_n + 7b_n + 1$
Since 1 can be written as $15 - 14$ we can write
$n + 1 = 5a_n + 7b_n + 15 - 14 = 5(a_n + 3) + 7(b_n - 2)$
Since a_n is an integer greater than or equal to 0, then $(a_n + 3)$ is also an integer greater than 0
$(b_n - 2)$ is only positive if b_n is greater or equal to 2.
There are therefore three situations that we need to consider: $b_n \geq 2$, $b_n = 1$, and $b_n = 0$.

i) $b_n \geq 2$
Let us define $a_{n+1} = a_n + 3$ and $b_{n+1} = b_n - 2$. Both are positive integers and $n + 1$ can be written as:
$n + 1 = 5a_{n+1} + 7b_{n+1}$
In this case, $p(n + 1)$ is true.

ii) $b_n = 1$
$n = 5a_n + 7$
$n + 1 = 5a_n + 8$
$n + 1 = 5a_n + 28 - 20$
$n + 1 = 5 \times (a_n - 4) + 7 \times 4$
$n + 1$ can be written as 7 times a positive integer 4 and 5 times $(a_n - 4)$.
Notice that since $n = 5a_n + 7$ and $n \geq 24$, therefore $5a_n \geq 17$. Since a_n is an integer,

we conclude that $a_n \geq 4$. Therefore $(a_n - 4) \geq 0$.
We can set $a_{n+1} = a_n - 4$ and $b_{n+1} = 4$; both are positive integers. We have, $n+1 = 5a_{n+1} + 7b_{n+1}$. In this case, $p(n+1)$ is true

iii) $b_n = 0$
$n = 5a_n$
$n+1 = 5a_n + 1$
$n+1 = 5a_n + 21 - 20$
$n+1 = 5 \times (a_n - 4) + 7 \times 3$
$n+1$ can be written as 7 times a positive integer 3 and 5 times $(a_n - 4)$.
Notice that since $n = 5a_n$ and $n \geq 24$, therefore $5a_n \geq 24$. Since a_n is an integer, we conclude that $a_n \geq 4$. Therefore $(a_n - 4) \geq 0$.
We can set $a_{n+1} = a_n - 4$ and $b_{n+1} = 3$; both are positive integers. We have, $n+1 = 5a_{n+1} + 7b_{n+1}$. In this case, $p(n+1)$ is true

In all cases, we have proven that $p(n+1)$ is true: the inductive step is true.

The principle of proof by mathematical induction allows us to conclude that $p(n)$ is true for all $n \geq 24$.

3) *Let u_n be the sequence of real numbers defined by*

$$\begin{aligned} u_0 &= 3 \\ u_{n+1} &= \frac{u_n - 2}{2u_n + 5} \end{aligned}$$

Show that:

$$u_n = \frac{9 - 8n}{3 + 8n}$$

Definitions:
Let:

$$\begin{aligned} LHS(n) &= u_n \\ RHS(n) &= \frac{9 - 8n}{3 + 8n} \\ P(n) : LHS(n) &= RHS(n) \end{aligned}$$

We want to Prove : $P(n)$ is true, for all $n \geq 1$.

- *Basis step:* We want to prove $P(1)$ is true.
  $LHS(1) = u_1 = \frac{u_0 - 2}{2u_0 + 5} = \frac{1}{11}$,
  and
  $RHS(1) = \frac{9-8}{3+8} = \frac{1}{11}$.
  Therefore, $LHS(1) = RHS(1)$: $P(1)$ is true.
- *Inductive step:* We want to show that $P(n) \to P(n+1)$, for $n \geq 1$.

We assume that $P(n)$ is true for a natural number $n \geq 1$, which means $LHS(n) = RHS(n)$. To prove that $P(n+1)$ is true, we prove that $LHS(n+1) = RHS(n+1)$. Let us compute $LHS(n+1)$:

$$\begin{aligned} LHS(n+1) &= u_{n+1} \\ &= \frac{u_n - 2}{2u_n + 5} \end{aligned}$$

As $LHS(n) = RHS(n)$, $u_n = \frac{9-8n}{3+8n}$. Therefore,

$$\begin{aligned} LHS(n+1) &= \frac{\frac{9-8n}{3+8n} - 2}{2\frac{9-8n}{3+8n} + 5} \\ &= \frac{9 - 8n - 2(3+8n)}{2(9-8n) + 5(3+8n)} \\ &= \frac{3 - 24n}{33 + 24n} \\ &= \frac{1 - 8n}{11 + 8n} \end{aligned}$$

and

$$\begin{aligned} RHS(n+1) &= \frac{9 - 8(n+1)}{3 + 8(n+1)} \\ &= \frac{1 - 8n}{11 + 8n} \end{aligned}$$

Therefore $LHS(n+1) = RHS(n+1)$, i.e. $P(n+1)$ is true.

According to the principle of mathematical induction, we can conclude that $u_n = \frac{9-8n}{3+8n}$ for all $n \geq 1$.

4) *Let f_n be the n-th Fibonacci number (note: Fibonacci numbers satisfy $f_0 = 0$, $f_1 = 1$ and $f_n + f_{n+1} = f_{n+2}$). Prove by induction that for all natural numbers $n \geq 1$, f_{3n} is even.*

Let us define: $P(n)$: f_{3n} is even

We want to Prove : $P(n)$ is true, for all $n \geq 1$. We use a proof by induction,

- *Basis step:* We want to prove $P(1)$ is true.
  $f_3 = f_2 + f_1 = 1 + 1 = 2$,
  which is even: $P(1)$ is true.
- *Inductive step:* We want to show that $P(n) \to P(n+1)$, for $n \geq 1$.

  We assume $P(n)$ to be true for a natural number $n \geq 1$, which means f_{3n} is even. There exists an integer k such that $f_{3n} = 2k$.

Then,

$$\begin{aligned}
f_{3(n+1)} &= f_{3n+3} \\
&= f_{3n+2} + f_{3n+1} \\
&= f_{3n+1} + f_{3n} + f_{3n+1} \\
&= f_{3n} + 2f_{3n+1} \\
&= 2k + 2f_{3n+1} \\
&= 2(k + f_{3n+1})
\end{aligned}$$

Since $k + f_{3n+1}$ is an integer, $f_{3(n+1)}$ is even, therefore. $P(n+1)$ is true.

According to the principle of mathematical induction, we can conclude that $P(n)$ is true for all $n \geq 1$.