Data, Logic, and Computing

ECS 17 (Winter 2025)

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Final: solutions

Part 1: Data *(10 questions, each 3 points; total 30 points)*

1) *Convert the binary number* $(1011010111001100)_2$ *to hexadecimal:*

   a) *#B5CC*
   b) *#B6CC*
   c) *#D6CC*
   d) *#D5CC*

2) *What is the largest signed integer that can be stored in 4 bits?*

   a) *15*
   b) *16*
   c) *7*
   d) *8*

   As the integer is signed, one bit is reserved for the sign. The largest number on the remaining 3 bits is $(111)_2$, i.e. 7.

3) $(1101)_2 + (1011)_2 =$

   a) *#18*
   b) *#1A*
   c) *#1C*
   d) *#1E*

   Note that:

   a) $(1101)_2 = (13)_{10}$
   b) $(1011)_2 = (11)_{10}$

   Therefore $(1101)_2 + (1011)_2 = (24)_{10} =$ #18.

4) *A camera captures video at 24 frames per second. Which of these moving objects can be properly recorded (Select all that apply)*

   a) *A hummingbird's wings flapping at 50 Hz*
   b) *A ceiling fan rotating at 0.8 revolutions per second*
   c) *A metronome moving at 30 beats per minute*
   d) *A strobe light flashing 10 times per second*

   As the sampling frequency is 24 Hz, the signal must have a frequency that is strictly smaller that 12 Hz:

   a) 50 Hz >12 Hz, so you can't see the wings properly
   b) 0.8 revolution per second is 0.8 Hz, which is < 12 Hz.
   c) 30 beats per minute is 0.5 beat per second, i.e. 0.5 Hz < 12 Hz
   b) 10 times per second is 10 Hz <12 Hz.

5) *A 1-minute video is captured with 30 frames per second, with each frame having a resolution of 1920×1080 pixels, and each pixel stored over 24-bit. What is the uncompressed file size in gigabytes (GB)? (Assume 1 GB = 10^9 bytes)*

   a) *8.9 GB*
   b) *11.2 GB*
   c) *3.7 GB*
   d) *5.6 GB*

   A 1 minute video is a 60 seconds video. 24 bit = 3 bytes.

   S $= 60 \times 30 \times 1920 \times 1080 \times 3 = 11197440000$ bytes, ≈ 11.2 GB.

6) *Which binary number comes right before the binary number 1000?*

   a) *0999*
   b) *0111*
   c) *0100*
   d) *1001*

7) *Decode the name whose ASCII representation is #54 #68 #6F #6D #61 #73*

   a) *Thomas*
   b) *THOMAS*
   c) *Thomes*
   d) *THOMES*

8) *Let A be the number with the hexadecimal representation #A and B the number whose hexadecimal representation is #18; which of these numbers X (in hexadecimal form) satisfies $X^2 - AX + B = 0$ (circle all that apply):*

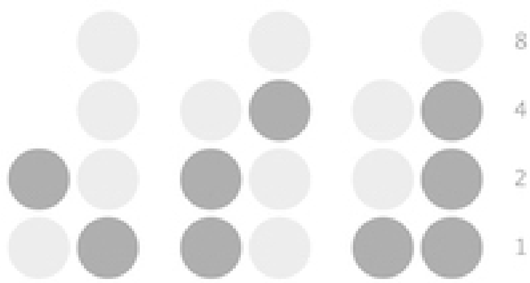
a) $\#C$
b) $\#4$
c) $\#2$
d) $\#6$

Note that:

a) $\#A = (10)_{10}$
b) $\#18 = (24)_{10}$

Therefore the quadratic equation is $X^2 - 10X + 24 = 0$, whose solutions are 4 and 6.

9) *What time is it on this digital clock (dark circle mean "on")?*



a) 21:24:16
b) 21:24:17
c) 21:34:16
d) 21:34:17

10) *Let $x$be a hexadecimal digit (i.e. a number in $\{0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F\}$ We know that $\#x3 + \#2x = \#BC$ (where # means that the numbers are given in hexadecimal format; for example, if $x = 1$, $\#x3 = \#13 = (19)_{10}$). Solve for x:*

a) *7*
b) *8*
c) *9*
d) *A*

The equation $\#x3+\#2x = \#BC$ is hexadecimal can be rewritten as $x\times 16+3+2\times 16+x = 11\times 16+12$, i.e. $17x = 153$, i.e., $x = 9$.

Part II *3 problems, each 10 points; total 30 points)*

1) *For each of the five propositions in the table below, indicates on the right if they are always tautologies or not (x is a real number, p and q are propositions, T is a tautology, and F is a contradiction).* (10 points)

Table 1: Propositional logic

Proposition	Tautology (Yes/ No)
$p \rightarrow p$	Yes: $p \rightarrow p \Leftrightarrow \neg p \vee p \Leftrightarrow T$.
$(p \vee q) \rightarrow (p \wedge q)$	No (table of truth)
$(p \vee F) \rightarrow p$	Yes (table of truth)
if $3+6=10$, then $25=16+10$	Yes! Note that p is false and therefore $p \rightarrow q$ is true.
$(p \wedge \neg q) \vee (\neg p \vee q)$	Yes (use table)

2) *Let p and q be two propositions. Use a truth table or logical equivalence to show that the proposition $\neg(p \rightarrow q) \leftrightarrow (p \wedge \neg q)$ is a tautology.*

Let us build the Truth table for $A = \neg(p \rightarrow q) \leftrightarrow (p \wedge \neg q)$

p	q	$\neg q$	$p \rightarrow q$	$\neg(p \rightarrow q)$	$p \wedge \neg q$	A
T	T	F	T	F	F	T
T	F	T	F	T	T	T
F	T	F	T	F	F	T
F	F	T	T	F	F	T

From Column 6, we can conclude that A is a tautology.

3) *Inspector Craig from Scotland Yard has been assigned a special mission: find a treasure supposedly hidden on an island where there is a single village. This village is inhabited only by truth-tellers and liars. Truth-tellers always tell the truth, while liars always lie. Inspector Craig meet two villagers, Daniel and Eve, and ask them about the treasure. Daniel says: "If I am a truth-teller, then the treasure is hidden in my house" while Eve says: "Daniel is a liar." Can you help inspector Craig and tell him if the treasure is in Daniel's house? Can you determine whether Daniel and Eve are truth-tellers or liars? Justify your answer. (10 points)*

Truth tellers are knights, liars are knaves. Daniel and Eve can be knights or knaves. There are therefore 4 cases to consider

Line number	Daniel	Eve	Daniel says	Eve says
1	Knight	Knight	?	F
2	Knight	Knave	?	F
3	Knave	Knight	T	T
4	Knave	Knave	T	T

Note that Daniel says a statement of the form $p \rightarrow q$, where

p: Daniel is a knight and

q: the treasure is an Daniel's house.

When Daniel is a knave, p is false, and therefore $p \rightarrow q$ is true. This would mean that if Daniel is a knave, he would tell the truth... this is not possible and therefore we can eliminate line 3 and 4.

We can eliminate also line 1, as Eve would be a knight that tells the truth.

Therefore, Daniel is a knight and Eve is a knave. Since Daniel is a knight, p is true and $p \rightarrow q$ is true, therefore q is true and the treasure is in Daniel's house.

Part III *4 problems, each 10 points; total 40 points)*

1) *Give a direct proof and an indirect proof of the proposition: if $2n^5 + n^2 + 4n + 2$ is odd, then n is odd, where n is a natural number).*

We want to prove an implication of the form $p \rightarrow q$ is true, with:

p: $2n^5 + n^2 + 4n + 2$ is odd $\neg p$: $2n^5 + n^2 + 4n + 2$ is even
q: n is odd $\neg q$: n is even.

We use both a direct proof and an indirect proof.

i) Let us show $p \rightarrow q$ using a direct proof:
Premise: p is true, i.e. $2n^5 + n^2 + 4n + 2$ is odd There exists an integer k such that let $2n^5 + n^2 + 4n + 2 = 2k + 1$,We get:

$$\begin{aligned} n^2 &= 2k - 2n^5 - 4n - 2 + 1 \\ &= 2(k - n^5 - 2n - 1) + 1 \end{aligned}$$

As $k - n^5 - 2n - 1$ is an integer, n^2 is odd. From the prayer, we know that if n^2 is odd, then n is odd. Therefore q is true.
We have shown that q is true, therefore $p \rightarrow q$ is true.

ii) Indirect proof: let us show $\neg q \rightarrow \neg p$.
Premise: $\neg q$ is true, i.e. that n is even.: there exists an integer k such that $n = 2k$. Therefore,

$$\begin{aligned} 2n^5 + n^2 + 4n + 2 &= 2 \times 2^5 \times k^5 + 4 \times k^2 + 4 \times 2 \times k + 2 \\ &= 2(2^5 \times k^5 + 2 \times k^2 + 4 \times k + 1) \end{aligned}$$

As $2^5 \times k^5 + 2 \times k^2 + 4 \times k + 1$ is an integer, $2n^5 + n^2 + 4n + 2$ is odd. We have shown that $\neg p$ is true, therefore $p \to q$ is true.

2) *Use induction to prove that any postage value of n cents can be made with only 4-cent stamps and 7-cent stamps, whenever $n \geq 18$, n natural number.*

Let $P(n)$ be the proposition that n cents can be made with only 4–cent and 7–cent stamps, when n is greater than or equal to 18.
This means that there exists two positive integers a_n and b_n such that $n = 4a_n + 7b_n$

a) Basis step: I want to prove that $P(18)$ is true
Note that: $18 = 4 \times 1 + 7 \times 2$
We can set $a_{18} = 1$ and $b_{18} = 2$. Both are positive integers, and $18 = 4a_{18} + 7b_{18}$. Therefore $P(18)$ is true.

b) Inductive Step
I want to show $P(n) \to P(n+1)$ whenever $n \geq 18$
Hypothesis: $P(n)$ is true and there exists two positive integers a_n and b_n such that $n = 4a_n + 7b_n$
Then:
$n + 1 = 4a_n + 7b_n + 1$
Since 1 can be written as $8 - 7$ we can write
$n + 1 = 4a_n + 7b_n + 2 \times 4 - 7 = 2(a_n + 2) + 7(b_n - 1)$
Since a_n is an integer greater than or equal to 0, then $(a_n + 2)$ is also an integer greater than 0
$(b_n - 1)$ is only positive if b_n is greater or equal to 1.
There are therefore two situations that we need to consider: $b_n \geq 1$, and $b_n = 0$.

i) $b_n \geq 1$
Let us define $a_{n+1} = a_n + 2$ and $b_{n+1} = b_n - 1$. Both are positive integers and $n+1$ can be written as:
$n + 1 = 4a_{n+1} + 7b_{n+1}$
In this case, $P(n+1)$ is true.

ii) $b_n = 0$
$n = 4a_n$
$n + 1 = 4a_n + 1$
$n + 1 = 4a_n + 21 - 20$
$n + 1 = 4 \times (a_n - 5) + 7 \times 3$
$n + 1$ can be written as 7 times a positive integer 3 and 4 times $(a_n - 5)$.
Notice that since $n = 4a_n$ and $n \geq 18$, $4a_n \geq 18$. Since a_n is an integer, we conclude that $a_n \geq 5$. Therefore $(a_n - 5) \geq 0$.
We can set $a_{n+1} = a_n - 5$ and $b_{n+1} = 3$; both are positive integers. We have, $n + 1 = 4a_{n+1} + 7b_{n+1}$. In this case, $P(n+1)$ is true

In all cases, we have proven that $P(n+1)$ is true: the inductive step is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n \geq 18$.

3) *Let fFn be the n-th Fibonacci number (note: Fibonacci numbers satisfy $F_0 = 0$, $F_1 = 1$ and $F_n + F_{n+1} = F_{n+2}$). Prove by induction that for all natural numbers $n \geq 1$, F_{5n} is a multiple of 5.*

Let us define: $P(n)$: F_{5n} is a multiple of 5, i.e. there exists an integer k such that $F_{5n} = 5k$

We want to Prove : $P(n)$ is true, for all $n \geq 1$. We use a proof by induction,

- *Basis step:* We want to prove $P(1)$ is true.
  $F_5 = F_4 + F_3 = F_3 + F_2 + F_2 + F_1 = 3F_2 + 2F_1 = 3 + 2 = 5$. Setting $k = 1$ (k is an integer), we have $F_5 = 5k$: P(1) is true.
- *Inductive step:* We want to show that $P(n) \rightarrow P(n+1)$, for $n \geq 1$.
  We assume $P(n)$ to be true for a natural number $n \geq 1$, which means F_{5n} is even. There exists an integer k such that $F_{5n} = 5k$.
  Then,

$$\begin{aligned}
F_{5(n+1)} &= F_{5n+5} \\
&= F_{5n+4} + F_{5n+3} \\
&= F_{5n+3} + F_{5n+2} + F_{5n+2} + F_{5n+1} \\
&= F_{5n+2} + F_{5n+1} + 2(F_{5n+1} + F_{5n}) + F_{5n+1} \\
&= F_{5n+1} + F_{5n} + F_{5n+1} + 2(F_{5n+1} + F_{5n}) + F_{5n+1} \\
&= 5F_{5n+1} + 3F_{5n} \\
&= 5F_{5n+1} + 3 \times 5k \\
&= 5(F_{5n+1} + 2k)
\end{aligned}$$

  Since $F_{5n+1} + 2k$ is an integer, $F_{5(n+1)}$ is a multiple of 5, therefore. $P(n+1)$ is true.

According to the principle of mathematical induction, we can conclude that $P(n)$ is true for all $n \geq 1$.

4) *Let a_n be the sequence of integers defined by*

$$\begin{aligned}
a_1 &= 5 \\
a_n &= 3a_{n-1} + 2
\end{aligned}$$

Show by induction that $a_n - 1$ is divisible by 4, $\forall n \geq 1$.

Definitions:
Let: $P(n)$: there exists and integer k such that $a_n - 1 = 4k$.

We want to Prove : $P(n)$ is true, for all $n \geq 1$.

- *Basis step:* We want to prove $P(1)$ is true.
  $a_1 - 1 = 5 - 1 = 4$. Setting $k = 1$, we have $a_1 - 1 = 4k$, where k is an integer: $P(1)$ is true.

- *Inductive step:* We want to show that $P(n) \rightarrow P(n+1)$, for $n \geq 1$.

  We assume that $P(n)$ is true for a natural number $n \geq 1$, which means there exists an integer k such that $a_n - 1 = 4k$. To prove that $P(n+1)$ is true, we need to find an integer l such that $a_{n+1} - 1 = 4l$

  $$\begin{aligned} a_{n+1} - 1 &= 3a_n + 1 \\ &= 3(a_n - 1) + 4 \\ &= 3 \times 4k + 4 \\ &= 4(3k + 1) \end{aligned}$$

  As $3k + 1$ is an integer, $a_{n+1} - 1$ is divisible by 4: $P(n+1)$ is true.

According to the principle of mathematical induction, we can conclude that $a_n - 1$ is divisible by 4 for all $n \geq 1$.