Data, Logic, and Computing

ECS 17 (Winter 2026)

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Final: solutions

Part 1: Data *(10 questions, each 3 points; total 30 points)*

1) $(11111)_2 + \#AE$

   a) $(205)_{10}$
   b) $(206)_{10}$
   c) $(204)_{10}$
   d) $(203)_{10}$

   $(11111)_2 = (31)_{10}$ and $\#AE = (174)_{10}$, hence the answer is $(205)_{10}$.

2) *2) What is the largest unsigned integer that can be stored in 3 bits?*

   a) *7*
   b) *8*
   c) *15*
   d) *16*

   The largest unsigned integer on n bits is $2^n - 1$. For $n = 3$, we find $2^3 - 1 = 7$.

3) $(88)_{10} + (10111)_2 =$

   a) *#6F*
   b) *#6E*
   c) *#F6*
   d) *#E6*

   Note that $(10111)_2 = (23)_{10}$, therefore $(88)_{10} + (10111)_2 = (111)_{10} = \#6F$.

4) *4) A smartphone camera is set to record slow-motion video at 60 frames per second. Which of the following continuous mechanical motions will be captured accurately without aliasing? (Select all that apply)*

a) *A car engine idling at 3000 RPM (Revolutions Per Minute)*

b) *A DJ's vinyl record spinning at 45 RPM*

c) *A power drill rotating at 15 Revolutions per second*

d) *An acoustic guitar's "A" string vibrating at 110 Hz*

As the sampling frequency is 60 Hz, the signal must have a frequency that is strictly smaller that 30 Hz:

a) 3000 RPM = 50 Hz > 30 Hz, so you can't see itproperly

b) 45 RPM = 0.75 Hz , which is < 30 Hz.

c) 15 revolutions per second, i.e. 15 Hz < 30 Hz

b) 110 Hz >30 Hz.

5) *A 1-minute video is captured with 30 frames per second, with each frame having a resolution of* 1080×1080 *pixels. How much space is used for each pixel if the video occupies approximately 4.2 gigabytes (GB)? (Assume 1 GB =* 10^9 *bytes)*

a) *8 bits*

b) *16 bits*

c) *32 bits*

d) *24 bits*

A 1 minute video is a 60 seconds video. If a pixels occupy x bytes,

S = $4200000000 = 60 \times 30 \times 1080 \times 1080 \times x$ bytes, hence $x = 2$ bytes, i.e., 16 bits.

6) *What is the binary representation of the number that is half the binary number (1000)2?*

a) $(0500)_2$

b) $(0100)_2$

c) $(0010)_2$

d) $(0004)_2$

$(1000)_2 = (8)_{10}$, hence its half is $(4)_{10} = (0100)_2$.

7) *The sum of the ASCII codes for the word ECS17 is*

a) #143

b) #144

c) #145

d) #142

The ASCII decimal values are E (69), C (67), S (83), 1 (49), and 7 (55). Their sum in decimal is 323. Converting 323 to hexadecimal gives #143.

8) *Let n be the number with the hexadecimal representation $\#A$ and m the number whose hexadecimal representation is $\#18$; which of these numbers X (in hexadecimal form) satisfies $X^2 + nX - m = 0$ (circle all that apply)::*

   a) $\#D$
   b) $\#4$
   c) $\#2$
   d) $\#E$

   Note that:

   a) $\#A = (10)_{10}$
   b) $\#18 = (24)_{10}$

   Therefore the quadratic equation is $X^2 + 10X - 24 = 0$, whose solutions are 2 and -12.

9) *9) A biologist wants to film a bee's wings flapping at 230 Hz and a moth's wings flapping at 40 Hz so that neither insect experiences the "wagon-wheel effect" on camera. Which of the following camera frame rates would successfully capture both? (Select all that apply)?*

   a) 24 frames per second (fps)
   b) 60 fps
   c) 240 fps
   d) 500 fps

   The largest frequency in the signal if $230Hz$, therefore the camera needs to take at least $2 \times 230 = 460$ frames per second.

10) *10) Let x be a hexadecimal digit (i.e. a number in $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}$. We know that $\#xA + \#Fx = \#1A4$ (where # means that the numbers are given in hexadecimal format; for example, if $x = 1$, $\#xA = \#1A = (26)_{10}$). Solve for x:*

   a) *7*
   b) A
   c) B
   d) C

   The equation $\#xA+\#Fx = \#1A4$ is hexadecimal can be rewritten as $x \times 16 + 10 + 15 \times 16 + x = 1 \times 16^2 + 10 \times 16 + 4$, i.e. $17x + 250 = 420$, i.e., $x = 10 = \#A$.

Part II *3 problems, each 10 points; total 30 points)*

1) *For each of the five propositions in the table below, indicates on the right if they are always tautologies or not (x is a real number, p and q are propositions, T is a tautology, and F is a contradiction).* (10 points)

Table 1: Propositional logic

Proposition	Tautology (Yes/ No)
$p \to \neg p$	No: $p \to \neg p \Leftrightarrow (\neg p \vee \neg p) \Leftrightarrow \neg p$. (False when p is True)
$(\neg p \wedge \neg q) \to (\neg p \vee \neg q)$	Yes (table of truth)
$(p \wedge q) \to (p \vee q)$	Yes (table of truth)
if $n^2 + n$ is odd, then $n = 5$	Yes! Note that p is false and therefore $p \to q$ is true.
$(p \wedge \neg p) \vee (\neg q \vee q)$	**Yes** (note that $\neg q \vee q$ is always true))

2) *Let p and q be two propositions. Use a truth table or logical equivalence to show that the proposition $[(p \to q) \to p] \to p$ is a tautology.*

We use the definition of implication ($A \to B \Leftrightarrow \neg A \vee B$) repeatedly:

$$\begin{aligned}
[(p \to q) \to p] \to p &\Leftrightarrow \neg[(p \to q) \to p] \vee p && \text{(Def. of Implication)} \\
&\Leftrightarrow \neg[\neg(\neg p \vee q) \vee p] \vee p && \text{(Def. of Implication)} \\
&\Leftrightarrow [(\neg p \vee q) \wedge \neg p] \vee p && \text{(De Morgan's Laws \& Double Negation)} \\
&\Leftrightarrow [(\neg p \wedge \neg p) \vee (q \wedge \neg p)] \vee p && \text{(Distributive Law)} \\
&\Leftrightarrow [\neg p \vee (\neg p \wedge q)] \vee p && \text{(Idempotent Law \& Commutativity)} \\
&\Leftrightarrow \neg p \vee p \vee (\neg p \wedge q) && \text{(Commutativity)} \\
&\Leftrightarrow \text{T} && \text{(Negation Law)}
\end{aligned}$$

Thus, the proposition is a tautology.

3) *Commander Ripley is exploring a derelict space station trying to locate the vital "override key." The station's computer system has rebooted, and now its android crew is split into two factions: Security Droids and Corrupted Droids. Security Droids always tell the truth, while Corrupted Droids always lie. Ripley encounters two androids, Alpha and Beta. Alpha says: "If I am a Security Droid, then the override key is in the Reactor Core." Beta says: "Alpha is a Corrupted Droid." Can you help Commander Ripley determine if the override key is in the Reactor Core? Can you figure out whether Alpha and Beta are Security Droids or Corrupted Droids? Justify your answer.(10 points)*

Security droids are knights, corrupted droids are knaves. Alpha and Beta can be knights or knaves. There are therefore 4 cases to consider.

Line number	Alpha	Beta	Alpha says	Beta says
1	Knight	Knight	?	F
2	Knight	Knave	?	F
3	Knave	Knight	T	T
4	Knave	Knave	T	T

We can eliminate line 3 and 4: Alpha would be a knave telling the truth. We can then eliminate line 1, as Beta would be a knight that lies.

Therefore, Alpha is a knight and Beta is a knave. Since Alpha is a knight, p is true and $p \to q$ is true, therefore q is true and the override key is in the reactor core.

Part III *4 problems, each 10 points; total 40 points)*

1) *Use a proof by induction to show that $3^n - 1$ is even, for all natural numbers $n \geq 1$..*

Let $P(n)$ be the proposition:

$$P(n) : 3^n - 1 \quad \text{is even}$$

We want to show that $P(n)$ is true for all natural numbers.

Base Case ($n = 1$): $3^1 - 1 = 2$. Since $2 = 2(1)$, it is even. The base case holds.
Inductive Hypothesis: Assume that for some arbitrary natural number $n \geq 1$, $3^k - 1$ is even. That is, $3^n - 1 = 2m$ for some integer m.
Inductive Step: We must show that $3^{m+1} - 1$ is even.

$$\begin{aligned} 3^{n+1} - 1 &= 3 \cdot 3^n - 1 \\ &= 3(2m + 1) - 1 && \text{(From IH: } 3^n = 2m + 1) \\ &= 6m + 3 - 1 \\ &= 6m + 2 \\ &= 2(3m + 1) \end{aligned}$$

Since m is an integer, $3m+1$ is an integer. Thus, $3^{n+1}-1$ is an even number. By the principle of mathematical induction, $3^n - 1$ is even for all natural numbers n.

2) *Prove by induction that for all natural numbers $n \geq 0$, F_{4n} is a multiple of 3.*

Let $P(n)$ be the proposition:

$$P(n) : F_{4n} \quad \text{is a multiple of 3}$$

We want to show that $P(n)$ is true for all natural numbers.

Base Case ($n = 1$): $F_{4(1)} = F_4 = F_3 + F_2 = 2F_2 + F_1 = 3$. Since $3 = 3(1)$, it is a multiple of 3. The base case holds.
Inductive Hypothesis: Assume that for some arbitrary natural number $n \geq 1$, F_{4n} is a multiple of 3. That is, $F_{4n} = 3m$ for some integer m.

Inductive Step: We must show that $F_{4(n+1)}$ is a multiple of 3. Note that $F_{4(n+1)} = F_{4n+4}$. Using the Fibonacci recurrence relation ($F_{n+2} = F_{n+1} + F_n$):

$$\begin{aligned} F_{4n+4} &= F_{4n+3} + F_{4n+2} \\ &= (F_{4n+2} + F_{4n+1}) + F_{4n+2} \\ &= 2F_{4n+2} + F_{4n+1} \\ &= 2(F_{4n+1} + F_{4n}) + F_{4n+1} \\ &= 3F_{4n+1} + 2F_{4n} \end{aligned}$$

Substitute the Inductive Hypothesis ($F_{4n} = 3m$):

$$F_{4n+4} = 3F_{4n+1} + 2(3m) = 3(F_{4n+1} + 2m)$$

Since F_{4n+1} and m are integers, $(F_{4n+1} + 2m)$ is an integer. Thus, $F_{4(n+1)}$ is a multiple of 3.

3) *Use induction to prove that any postage value of n cents can be made with only 3-cent stamps and 8-cent stamps, whenever $n \geq 15$, n natural number.*

Let $P(n)$ be the proposition that n cents can be made with only 3–cent and 8–cent stamps, when n is greater than or equal to 15.
This means that there exists two positive integers a_n and b_n such that $n = 3a_n + 8b_n$

a) Basis step: I want to prove that $P(15)$ is true
Note that: $15 = 3 \times 5$
We can set $a_{15} = 5$ and $b_{15} = 0$. Both are positive integers, and $15 = 3a_{15} + 8b_{15}$. Therefore $P(15)$ is true.

b) Inductive Step
I want to show $P(n) \to P(n+1)$ whenever $n \geq 15$
Hypothesis: $P(n)$ is true and there exists two positive integers a_n and b_n such that $n = 3a_n + 8b_n$
Then:
$n + 1 = 3a_n + 8b_n + 1$
Since 1 can be written as $9 - 8$ we can write
$n + 1 = 3a_n + 8b_n + 9 - 8 = 3(a_n + 3) + 8(b_n - 1)$
Since a_n is an integer greater than or equal to 0, $(a_n + 3)$ is also an integer greater than 0
$(b_n - 1)$ is only positive if b_n is greater or equal to 1.
There are therefore two situations that we need to consider: $b_n \geq 1$, and $b_n = 0$.

i) $b_n \geq 1$
Let us define $a_{n+1} = a_n + 3$ and $b_{n+1} = b_n - 1$. Both are positive integers and $n+1$ can be written as:
$n + 1 = 3a_{n+1} + 8b_{n+1}$
In this case, $P(n+1)$ is true.

ii) $b_n = 0$
$n = 3a_n$

$n+1 = 3a_n + 1$
$n+1 = 3a_n + 16 - 15$
$n+1 = 3 \times (a_n - 5) + 8 \times (2)$
$n+1$ can be written as 3 times $(a_n - 5)$ abd 8 times a positive integer 2.
Notice that since $n = 3a_n$ and $n \geq 15$, $3a_n \geq 15$. Since a_n is an integer, we conclude that $a_n \geq 5$. Therefore $(a_n - 5) \geq 0$.
We can set $a_{n+1} = a_n - 5$ and $b_{n+1} = 2$; both are positive integers. We have, $n+1 = 3a_{n+1} + 8b_{n+1}$. In this case, $P(n+1)$ is true

In all cases, we have proven that $P(n+1)$ is true: the inductive step is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n \geq 15$.

4) *Let $a_{n+1} = \frac{a_n}{\sqrt{a_n^2+1}}$ for $n \geq 1$ with $a_1 = 1$. Show by induction that $a_n = \frac{1}{\sqrt{n}}$ for all $n \geq 1$.*

Let $P(n)$ be the proposition:

$$P(n) : a_n = \frac{1}{\sqrt{n}}$$

We want to show that $P(n)$ is true for all natural numbers.

We want to Prove : $P(n)$ is true, for all $n \geq 1$. We use a proof by induction, **Base Case ($n = 1$):** $a_1 = \frac{1}{\sqrt{1}} = 1$. This matches the given initial condition $a_1 = 1$. The base case holds.
Inductive Hypothesis: Assume that for some arbitrary natural number $n \geq 1$, $a_n = \frac{1}{\sqrt{n}}$.
Inductive Step: We must show that $a_{n+1} = \frac{1}{\sqrt{n+1}}$. Substitute the Inductive Hypothesis into the recurrence relation:

$$\begin{aligned} a_{n+1} &= \frac{a_n}{\sqrt{a_n^2+1}} \\ &= \frac{\frac{1}{\sqrt{n}}}{\sqrt{\left(\frac{1}{\sqrt{n}}\right)^2+1}} \\ &= \frac{\frac{1}{\sqrt{n}}}{\sqrt{\frac{1}{n}+1}} \\ &= \frac{\frac{1}{\sqrt{n}}}{\sqrt{\frac{1+n}{n}}} \\ &= \frac{1}{\sqrt{n}} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} \\ &= \frac{1}{\sqrt{n+1}} \end{aligned}$$

This is exactly what we needed to show. By mathematical induction, $a_n = \frac{1}{\sqrt{n}}$ for all natural numbers n.