Data, Logic, and Computing

ECS 17 (Winter 2025)

Patrice Koehl
koehl@cs.ucdavis.edu

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Midterm 2: solutions

Exercise 1 *(1 question, 10 points)*

Let a, b, and c be 3 integers. Using the method of proof of your choice, show that if abc is even, then at least one of a, b, or c must be even.

We want to prove an implication of the form $p \rightarrow q$ is true, with:

p: abc is even	$\neg p$: abc is odd
q: a is even or b is even or c is even	$\neg q$: a and b and c are odd.

We use an indirect proof: we show that $\neg q \rightarrow \neg p$ is true.

Let us assume that $\neg q$ is true, namely that a, b, and c are odd. There exists three integers l, m, and n such that:

$$\begin{aligned} a &= 2l+1 \\ b &= 2m+1 \\ c &= 2n+1. \end{aligned}$$

Then,

$$\begin{aligned} abc &= (2l+1)(2m+1)(2n+1) \\ &= (4lm+2l+2m+1)(2n+1) \\ &= 8lmn+4lm+4ln+2l+4mn+2m+2n+1 \\ &= 2(4lmn+2lm+2ln+2mn+l+m+n)+1. \end{aligned}$$

As l, m, and n are integers, $4lmn+2lm+2ln+2mn+l+m+n$ is an integer. Therefore abc is odd.

Therefore $\neg p$ is true when $\neg q$ is true. the proposition $\neg q \rightarrow \neg p$ is true and, by equivalence, $p \rightarrow q$ is true.

Exercise 2 *(1 question, 10 points)*

Let n be an integer. Give an indirect proof of the proposition, if $n^2 + 8n$ is odd then $5n$ is odd.

We want to prove an implication of the form $p \to q$ is true, with:

p: $n^2 + 8n$ is odd	$\neg p$: $n^2 + 8n$ is even
q: $5n$ is odd	$\neg q$: $5n$ is even

We use an indirect proof, i.e. we show $\neg q \to \neg p$ is true.

Let us assume that $\neg q$ is true, i.e. that $5n$ is even. There exists and integer k such that $5n = 2k$. Therefore,

$$\begin{aligned} n^2 + 8n &= n^2 + n + 5n + 2n \\ &= n(n+1) + 2k + 2n \end{aligned}$$

We have shown in class that $n(n+1)$ is always even, when n is an integer. Therefore there exists an integer l such that $n(n+1) = 2l$. Therefore,

$$\begin{aligned} n^2 + 8n &= 2l + 2k + 2n \\ &= 2(k + l + n) \end{aligned}$$

As k, n, and l are integers, $k + l + n$ is an integer. Therefore $n^2 + 8n$ is even.

We have shown that $\neg p$ is true when $\neg q$ is true: the proposition $\neg q \to \neg p$ is true and, by equivalence, $p \to q$ is true.

Exercise 3 *(1 question, 10 points)*

Let n be an integer. Use a proof by contradiction to show that $\frac{n^3+n^2+3}{2n^2+6}$ is not an integer.

Let:
P: $\frac{n^3+n^2+3}{2n^2+6}$ is not an integer

We use a proof by contradiction. We **assume** that P is false, i.e. we assume that $\frac{n^3+n^2+3}{2n^2+6}$ is an integer. Let us name this integer as k. We have:

$$\frac{n^3 + n^2 + 3}{2n^2 + 6} = k$$

which we rewrite as:

$$n^3 + n^2 + 3 = k(2n^2 + 6)$$

Let $LHS = n^3 + n^2 + 3$ and $RHS = k(2n^2 + 6)$. Notice that:

$$\begin{aligned} LHS &= n(n(n+1)) + 3 \\ &= 2ln + 2 + 1 \\ &= 2(ln + 1) + 1 \end{aligned}$$

where we have used that $n(n+1)$ is even, i.e. there exists an integer l such that $n(n+1) = 2l$. Since $ln + 1$ is an integer, LHS is odd. Conversely,

$$RHS = 2(k(n^2 + 3))$$

As k and n are integers, $k(n^2+3)$ is an integer and therefore RHS is even.

Under the assumption that P is false, we find that $LHS = RHS$ with LHS odd and RHS even. Since an even number cannot be equal to an odd number, we have reached a contradiction. Therefore the assumption that P is false, is false, i.e. P is true.

Exercise 4 *(1 question, 10 points)*

Let n be an integer. Use a direct proof to show that if n is odd, then there exists an integer m such that $n^2 = 8m+1$.

We want to prove an implication of the form $p \to q$ is true, with:

p: n is odd

q: There exists an integer m such that $n^2 = 8m+1$.

We use a direct proof. We assume that p is true.

Since p is true, n is odd: there exists an integer k such that $n = 2k+1$. Therefore,

$$\begin{aligned} n^2 &= 4k^2 + 4k + 1 \\ &= 4k(k+1) + 1. \end{aligned}$$

As k is an integer, $k(k+1)$ is even. Therefore there exists an integer l such that $k(k+1) = 2l$. Therefore

$$n^2 = 8l + 1.$$

We have found an integer m ($m = l$) such that $n^2 = 8m+1$: q is true.

We have shown that q is true when p is true: the proposition $p \to q$ is true.

Exercise 5 *(1 question, 10 points)*

Let n be an integer. Show that $4n+3$ is not a perfect square. (An integer a is a perfect square if and only if there exists an integer b such that $a = b^2$).

Let:

P: $4n+3$ is not a perfect square.

We use a proof by contradiction. We **assume** that P is false, i.e. we assume that $4n+3$ is a perfect square. Then there exists an integer k such that:

$$4n + 3 = k^2$$

Since $4n+3$ is odd, k^2 is odd and therefore k is odd. We write it at $k = 2l+1$ where l is an integer. Then,

$$\begin{aligned} 4n+3 &= (2l+1)^2 \\ &= 4k^2 + 4k + 1 \end{aligned}$$

We rewrite it as:

$$2 = 4k^2 + 4k - 4n$$

After dividing by 2,

$$
\begin{aligned}
1 &= 2k^2 + 2k - 2n \\
1 &= 2(k^2 + k - n).
\end{aligned}
$$

As k and n are integers, $k^2 + k - n$ is an integer and therefore $2(k^2 + k - n)$ is even. But 1 is odd. Since an even number cannot be equal to an odd number, we have reached a contradiction. Therefore the assumption that P is false, is false, i.e. P is true.