Data, Logic, and Computing

ECS 17 (Winter 2022)

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February 20, 2026

Midterm 2: solutions

Part I: logic (1 question, 10 points)

Using a truth table, establish if the poposition below if it is a tautology, a contradiction or neither.
$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$

p	q	$p \rightarrow q$	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$	$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Therefore the proposition $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$ is a tautology.

Part II: proofs (4 questions, each 10 points; total 40 points)

1) *Prove that if $7n^2 + 4$ is even, then $n^2 + 2$ is even, where n is a natural number.*

We want to prove an implication of the form $p \rightarrow q$ is true, with:

$$p : 7n^2 + 4 \text{ is even} \quad \neg p : 7n^2 + 4 \text{ is odd}$$
$$q : n^2 + 2 \text{ is even} \quad \neg q : n^2 + 2 \text{ is odd}$$

We use a direct proof:

We assume that p is true, i.e. that $7n^2 + 4$ is even. There exists an integer k such that $7n^2 + 4 = 2k$. Let us first rewrite:

$$7n^2 + 4 \quad = \quad 6n^2 + 2 + n^2 + 2$$

We then have:

$$2k = 6n^2 + 2 + n^2 + 2$$

and therefore

$$\begin{aligned} n^2+2 &= 2k-6n^2-2 \\ &= 2(k-3n^2-1) \end{aligned}$$

As $k-3n^2-1$ is an integer, n^2+2 is even, and therefore q is true.

We have shown that $p \to q$ is true using a direct proof.

2) *Let n be a natural number. Use a direct proof to show that if $6n^2+3n+11$ is odd, then n is even.*

We want to prove an implication of the form $p \to q$ is true, with:

$$p : 6n^2+3n+11 \text{ is odd}$$
$$q : n \text{ is even}$$

We use a direct proof:

We assume that p is true, i.e. that $6n^2+3n+11$ is odd. There exists an integer k such that $6n^2+3n+11 = 2k+1$, which we rewrite:

$$\begin{aligned} 6n^2+3n+11 &= 2k+1 \\ n+6n^2+2n+11 &= 2k+1 \\ n &= 2k+1-6n^2-2n-11 \\ n &= 2k-6n^2-2n-10 \\ n &= 2(k-3n^2-n-5) \end{aligned}$$

As $k-3n^2-n-5$ is an integer, n is even, and therefore q is true.

We have shown that $p \to q$ is true using a direct proof.

3) *Let n be a natural number. Use a proof by contradiction to show that $\sqrt{4n+2}$ is not an integer.*

We use a proof by contradiction. If p is the proposition, "$\sqrt{4n+2}$ is not an integer", then $\neg p$ is the proposition "$\sqrt{4n+2}$ is an integer".

We assume that $\neg p$ is true. There exists an integer k such that $\sqrt{4n+2} = k$. After squaring,

$$\begin{aligned} k^2 &= 4n+2 \\ &= 2(2n+1) \end{aligned}$$

As $2n+1$ is an integer, k^2 is even, and therefore k is even (prayer).

As k is even, there exists an integer l such that $k = 2l$. Replacing above,

$$\begin{aligned} (2l)^2 &= 4n+2 \\ 4l^2 &= 4n+2 \end{aligned}$$

After division by 2,

$$2l^2 = 2n + 1$$

As l is an integer, $2l^2$ is even. Similarly, as n is an integer, $2n + 1$ is odd. This leads to a contradiction, as an even number can't be equal to an odd number. Therefore it is false that $\neg p$ is false, and p is true.

We have shown p is true using a proof by contradiction.

4) *Let a and b be 2 integers. Use a direct proof to show that if $a^2 + b^2$ is even, then $a^2 - b^2$ is even.*

We want to prove an implication of the form $p \to q$ is true, with:

$$p : a^2 + b^2 \text{ is even}$$
$$q : a^2 - b^2 \text{ is even}$$

We use a direct proof:

We assume that p is true, i.e. that $a^2 + b^2$ is even. There exists an integer k such that $a^2 + b^2 = 2k$. Let us first rewrite:

$$\begin{aligned} a^2 - b^2 &= a^2 + b^2 - 2b^2 \\ &= 2k - 2b^2 \\ &= 2(k - b^2) \end{aligned}$$

As $k - b^2$ is an integer, $a^2 - b^2$ is even, and therefore q is true.

We have shown that $p \to q$ is true using a direct proof.

Part III: knights and knaves (1 question, 10 points)

A very special island is inhabited only by Knights and Knaves. Knights always tell the truth, while Knaves always lie. You meet three inhabitants: Alex, John and Sally. Alex says, "John is a Knight if and only if Sally is a Knave". John says, "If Sally is a Knight, then Alex is a Knight". Can you find what Alex, John, and Sally are? Explain your answer.

Let us build the table for the possible options for Alex, John, and Sally. We then check the validity of the two statements, and finally check the consistency of the truth values for those statements with the nature of Alex and John.

Line	Alex	John	Sally	Alex says	John says	Compatibility
1	Knight	Knight	Knight	F	T	No: Alex would be a Knight who lies
2	Knight	Knight	Knave	T	T	Yes
3	Knight	Knave	Knight	T	T	No, John would be a Knave who tells the truth
4	Knight	Knave	Knave	F	T	No, John would be a Knave who tells the truth
5	Knave	Knight	Knight	F	F	No, John would be a Knight who lies
6	Knave	Knight	Knave	T	T	No, Alex would be a Knave who tells the truth
7	Knave	Knave	Knight	T	F	No, Alex would be a Knave who tells the truth
8	Knave	Knave	Knave	F	T	No, John would be a Knave who tells the truth

Therefore Alex and John are Knights and Sally is a Knave.