ECS 17: Data, Logic, and Computing
Midterm 2 - makeup
February 23, 2022

Notes:
1) Midterm is open book, open notes...
2) You have 50 minutes, no more: I will strictly enforce this.
3) You can answer directly on these sheets (preferred), or on loose paper.
4) Please write your name at the top right of at least the first page that you turn in!
5) Please, check your work!

Proofs

Exercise 1 (2 questions, each 10 points; total 20 points)
Let $n$ be an integer. Show that if $4n^2 + 3n + 9$ is odd, then $n$ is even, using
(a) An indirect proof

Let $n$ be an integer.

Let $p : 4n^2 + 3n + 9$ is odd

Let $\neg p : 4n^2 + 3n + 9$ is even

9: $n$ is even

$\neg 9 : n$ is odd.

We show $p \rightarrow q$ using an indirect proof, i.e. we show $\neg q \rightarrow \neg p$.

Hypothesis: $\neg q$ is true. There exists an integer $k$ such that $n = 2k + 1$.

$4n^2 + 3n + 9 = 4(2k+1)^2 + 3(2k+1) + 9$

$= 16k^2 + 16k + 4 + 6k + 3 + 9$

$= 16k^2 + 22k + 16$

$= 8k^2 + 11k + 8$ is an integer, $4n^2 + 3n + 9$ is even

$\neg p$ is true.
(b) A direct proof (hint: find an expression for $n$)

Direct proof: There exists an integer $k$ such that $4m^2 + 3m + 9 = 2k + 1$

There are:

$m = -2n - 4m^2 + 2k - 8$

$n = 2(-m - 2n^2 + k - 4)$

$m - 2n^2 + k - 4$ is an integer, therefore $n$ is even. $q$ is true.
Exercise 2 (2 questions, each 10 points; total 20 points)

1) Let \( n \) be an integer. Use a direct proof to show that if \( n^3 \) is even, then \( n \) is even. *(Hint: add \( n^2 \) to \( n^3 \))*

Let \( p: \) \( m^3 \) is even

\( q: \) \( n \) is even.

Direct proof. Since \( m^3 \) is even, there exists an integer \( k \) such that \( m^3 = 2k \).

Using the hint:

\[
\begin{align*}
    m^3 + n^2 &= m^2 + 2k \\
    m^2 (m+1) &= m^2 + 2k
\end{align*}
\]

Therefore:

\[
m^2 = m\left\lfloor \frac{m(m+1)}{2} \right\rfloor - 2k
\]

Based on the appendix, \( m(m+1) \) is always even.

Therefore:

\[
m^2 = 2mP - 2k
\]

\( mP-k \) is an integer. Therefore \( n^2 \) is even.

Based on the appendix, if \( n^2 \) is even then \( n \) is even.

This concludes the proof.
2) Show that $\sqrt{2}$ is irrational

Proof by contradiction. Let us assume $\sqrt{2}$ is rational. There exists a pair of integers $(a, b)$ with $b \neq 0$ and without common factor such that

$$\sqrt{2} = \frac{a}{b}$$

Taking power 3,

$$2 = \frac{a^3}{b^3}$$

$$a^3 = 2b^3$$

As $b^3$ is an integer, $a^3$ is even. According to the exercise above, then $a$ is even. There exists an integer $k$ such that $a = 2k$, replacing a bare

$$\sqrt{2k^3} = 2 \sqrt{k^3}$$

$$b^3 = 4k^3 = 2(2k^3)$$

$2k^3$ is an integer, therefore $b^3$ is even. According to the exercise above, then $b$ is even. But $a$ and $b$ would be even, and without a common factor: contradiction.

Therefore $\sqrt{2}$ is irrational.
Exercise 3 (1 question, 10 points)

Let \( a \) and \( b \) be two odd integers, with \( a < b \). Show that there exits an integer \( c \) such that \( |a-c|=|b-c| \)

where \(| | \) stands for the absolute value, defined as
\[
|y| = \begin{cases} y & \text{if } y \geq 0 \\ -y & \text{if } y < 0 \end{cases}
\]

for any integer \( y \).

We know \( a < b \). There are therefore 3 options for \( c \), \( c < a < b \), \( a < c < b \), \( a < b < c \)

\[ a < b < c \]

Case 1: \( c < a < b \). Then \( |a-c| = a-c \) and \( |b-c| = b-c \)

The equation becomes \( a-c = b-c \), \( a = b \)

\( \rightarrow \) no solution.

Case 2: \( a < b < c \). Then \( |a-c| = c-a \) and \( |b-c| = c-b \)

The equation becomes \( c-a = c-b \), \( 0 = b-a \)

\( \rightarrow \) no solution.

Case 3: \( a < c < b \). Then \( |a-c| = c-a \) and \( |b-c| = b-c \)

The equation becomes \( c-a = b-c \)

\[ 2c = a + b \]

Note that \( a \) is odd: there exists \( k \) such that \( a = 2k+1 \)

\( b \) is odd: there exists \( t \) such that \( b = 2t+1 \)

\[ c = \frac{a+b}{2} = \frac{2k+2t+2}{2} = k+t+1 \]

\( c \) is an integer.

The only solution is \( c = \frac{a+b}{2} \).
Exercise 4 (1 question, 10 points)
Let \(a, b,\) and \(c\) be three consecutive integers. Show that if \(a \neq 1\) and \(a \neq 3,\) then \(a^2 + b^2 \neq c^2.\)

\[ p: a \neq 1 \text{ and } a \neq 3 \quad q: a^2 + b^2 = c^2 \quad \neg p: a = -1 \text{ or } a = 3 \quad \neg q: a^2 + b^2 \neq c^2 \]

We use an indirect proof. We assume \(\neg q\) is true, i.e.

\[ a^2 + b^2 = c^2 \]

\(a, b,\) and \(c\) are consecutive: \(b = a + 1\) and \(c = a + 2\)

Thus we have:

\[ a^2 \quad (a+1)^2 = (a+2)^2 \]

\[ a^2 + a^2 + 2a + 1 = a^2 + 4a + 4 \]

\[ a^2 - 2a - 3 = 0 \]

\[ (a+1)(a-3) = 0 \]

Thus we have \(a = -1, a = 3, \neg p\) is true.

Exercise 5 (1 question, 10 points)
Let \(m, n,\) and \(p\) be integers. Show that if \(m+n\) and \(n+p\) are even, then \(m+p\) is even.

We use a direct proof.

\(m+n\) is even: \(\exists k \in \mathbb{Z}, m+n = 2k\)

\(n+p\) is even: \(\exists f \in \mathbb{Z}, n+p = 2f\)

Summing these 2 relations:

\[ m+2n+p = 2k + 2f \]

\[ m+p = -2n + 2k + 2f \]

\[ -m + k + f \text{ is an integer, therefore } m+p \text{ is even.} \]