Midterm 2: solutions

Exercise 1 (1 question, 10 points)

Let $a$ and $b$ be two real numbers, with $a \neq 0$ and $b \neq 0$. Use a proof by contradiction to show that if $ab > 0$, then $\frac{a}{b} + \frac{b}{a} \geq 2$.

Let:

$p$: $ab > 0$

$q$: $\frac{a}{b} + \frac{b}{a} \geq 2$

and let $A$ be the proposition $p \rightarrow q$. We want to show that $A$ is true. We use a proof by contradiction, i.e. we suppose that what we want to show is false, namely that $A$ is not true, i.e. $p$ is true AND $q$ is false.

$p$ is true: $ab > 0$. Similarly, as $q$ is false, $\frac{a}{b} + \frac{b}{a} < 2$. As $ab > 0$, we can multiply this inequality by $ab$ without changing its sense; we get:

$$a^2 + b^2 < 2ab$$

which gives

$$a^2 + b^2 - 2ab < 0$$

i.e.

$$(a - b)^2 < 0$$

However, $(a - b)^2$ is a square, and therefore $(a - b)^2 \geq 0$. we have reached a contradiction. The proposition $A$ is therefore true.

Exercise 2 (2 questions, each 10 points; total 20 points)

1) Let $a$ and $b$ be two integers. Show that if $a + b\sqrt{2} = 0$, then $a = 0$ and $b = 0$.

Let:

$p$: $a + b\sqrt{2} = 0$
q: $a = 0$ and $b = 0$

and let $A$ be the proposition $p \rightarrow q$. We want to show that $A$ is true. We use a proof by contradiction, i.e. we suppose that what we want to show is false, namely that $A$ is not true, i.e. $p$ is true AND $q$ is false.

As $q$ is false, either $a \neq 0$ or $b \neq 0$. As $b$ may be 0, we look at the two corresponding cases:

a) $b = 0$. Since $p$ is true, we get $a = 0$. But this is in contradiction with $\neg q$: we have reached a contradiction.

b) $b \neq 0$. We can then divide by $b$ in $p$ and we obtain:

$$\sqrt{2} = \frac{-a}{b}$$

As $-a$ and $b$ are integers, with $b \neq 0$, this suggests that $\sqrt{2}$ is rational... but we know that this is not true. We have reached a contradiction.

In all cases, we have reached a contradiction. The proposition $A$ is therefore true.

2) Let $m$, $n$, $p$, and $q$ be four integers. Show that $m + n\sqrt{2} = p + q\sqrt{2}$, if and only if $m = p$ and $n = q$.

Let:

P: $m + n\sqrt{2} = p + q\sqrt{2}$

Q: $m = n$ and $p = q$

and let $A$ be the proposition $P \leftrightarrow Q$. We want to show that $A$ is true. We prove both $P \rightarrow Q$ and $Q \rightarrow P$.

a) Lest us prove $P \rightarrow Q$. We use a direct proof.

We assume $P$ is true, namely that

$$m + n\sqrt{2} = p + q\sqrt{2}$$

which we rewrite as:

$$(m - p) + (n - q)\sqrt{2} = 0$$

As $m - p$ and $n - q$ are both integers, based on part 1, we find that $m - p = 0$ and $n - q = 0$, namely $m = p$ and $n = q$. Therefore $Q$ is true, and consequently $P \rightarrow Q$ is true.

b) Lest us prove $Q \rightarrow P$. We use a direct proof.

As $Q$ is true, $m = p$ and $n = q$. Then $m + n\sqrt{2} = p + q\sqrt{2}$, namely $P$ is true. Therefore $Q \rightarrow P$ is true.

We have shown that $P \rightarrow Q$ and $Q \rightarrow P$ are true. Therefore $A$ is true.
Exercise 3 (1 question, 10 points)

Let $x$ be a real number. Show that $|x - 1| \leq x^2 - x + 1$, where $||$ is the absolute value.

Let:

$P$: $|x - 1| \leq x^2 - x + 1$

where $x$ is a real number. Let us define $A(x) = |x - 1|$ and $B(x) = x^2 - x + 1$. To show that $P$ is true, we use a proof by case:

a) $x \geq 1$. Then,

$$A(x) = x - 1$$
$$B(x) = x^2 - x + 1$$

Therefore

$$A(x) - B(x) = x - 1 - x^2 + x - 1$$
$$= -(x^2 - 2x + 1) - 1$$
$$= -(x - 1)^2 - 1$$
$$< 0$$

Therefore $A(x) \leq B(x)$ and $P$ is true.

b) $x < 1$. Then,

$$A(x) = -x + 1$$
$$B(x) = x^2 - x + 1$$

Therefore

$$A(x) - B(x) = -x + 1 - x^2 + x - 1$$
$$= -x^2$$
$$< 0$$

Therefore $A(x) \leq B(x)$ and $P$ is true.

In all cases, $P$ is true.

Exercise 4 (1 question, 10 points)

Let $a$ and $b$ be two integers. Show that if $(a^2 + b^2)^2$ is even, then $a + b$ is even.

Let:

$p$: $(a^2 + b^2)^2$ is even
$q$: $a + b$ is even

and let $A$ be the proposition $p \rightarrow q$. We want to show that $A$ is true. We use a direct proof.

We assume that $(a^2 + b^2)^2$ is even. Let $n = a^2 + b^2$; $n$ is an integer. The assumption is therefore that $n^2$ is even. We know then that $n$ is even. Therefore $a^2 + b^2$ is even, but this leads to $a + b$ is even, using the theorems we can assume to be true. Therefore $q$ is true, and $A$ is true by direct proof.
Exercise 5 (1 question, 10 points)

Let $a$ and $b$ be two real numbers. Show that if $a \neq -1$ and $b \neq -1$, then $a + b + ab \neq -1$.

Let:
- $p$: $a \neq -1$ and $b \neq -1$
- $q$: $a + b + ab \neq -1$

and let $A$ be the proposition $p \rightarrow q$. We want to show that $A$ is true. We use an indirect proof. We need to define first:
- $\neg p$: $a = -1$ or $b = -1$
- $\neg q$: $a + b + ab = -1$

We assume $\neg q$. Therefore $a + b + ab = -1$. Using the hint, we find $(a + 1)(b + 1) - 1 = -1$, i.e. $(a + 1)(b + 1) = 0$. This leads to $a = -1$ or $b = -1$, namely $\neg p$ is true. Therefore $A$ is true by indirect proof.

Exercise 6 (1 question, 10 points)

Let $a$ and $b$ be two integers. Show that $a^2 - 4b \neq 2$.

Let:
- $P$: $a^2 - 4b \neq 2$ where $a$ and $b$ are two integers. To show that $P$ is true, we use a proof by contradiction, namely we assume $P$ is false:

$$a^2 - 4b = 2$$

We get:

$$a^2 = 2 + 4b = 2(1 - 2b)$$

As $1 - 2b$ is an integer, $a^2$ is even, and therefore $a$ is even. There exists an integer $k$ such that $a = 2k$. Replacing above, we get:

$$
\begin{align*}
(2k)^2 - 4b &= 2 \\
4k^2 - 4b &= 2
\end{align*}
$$

After division by 2,

$$2k^2 - 2b = 1$$

$2k^2 - 2b = 2(k^2 - b)$ and since $k^2 - b$ is an integer, $2k^2 - 2b$ is even. However, 1 is odd: we have reached a contradiction. Therefore $P$ is true.