Data, Logic, and Computing

ECS 17 (Winter 2024)

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February 25, 2024

Midterm 2: solutions

Part I: logic (2 questions, each 10 points; total 20 points)

Using truth tables, establish for each of the two propositions below if it is a tautology, a contradiction or neither.

1) $(p \leftrightarrow q) \leftrightarrow (\neg p \leftrightarrow \neg q)$

p	q	$p \leftrightarrow q$	$\neg p$	$\neg q$	$\neg p \leftrightarrow \neg q$	$(p \leftrightarrow q) \leftrightarrow (\neg p \leftrightarrow \neg q)$
T	T	T	F	F	T	T
T	F	F	F	T	F	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T

The proposition is a tautology.

2) $(p \rightarrow (q \wedge r)) \vee ((p \wedge q) \rightarrow r)$

p	q	r	$q \wedge r$	$p \rightarrow (q \wedge r)$	$p \wedge q$	$(p \wedge q) \rightarrow r$	$(p \rightarrow (q \wedge r)) \vee ((p \wedge q) \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	F	F	T	T
T	F	F	F	F	F	T	T
F	T	T	T	T	F	T	T
F	T	F	F	T	F	T	T
F	F	T	F	T	F	T	T
F	F	F	F	T	F	T	T

The proposition is not a tautology.

Part II: proofs (5 questions, each 10 points; total 50 points)

1) *Prove that if $7n^2 + 4$ is even, then n is even, where n is a natural number.*

We want to prove an implication of the form $p \to q$ is true, with:

$$\begin{array}{cc} p : 7n^2 + 4 \text{ is even} & \neg p : 7n^2 + 4 \text{ is odd} \\ q : n \text{ is even} & \neg q : n \text{ is odd} \end{array}$$

We use an indirect proof:

We assume that $\neg q$ is true, i.e. that n is odd. There exists an integer k such that $n = 2k+1$. Then,

$$\begin{aligned} 7n^2 + 4 &= 7(2k+1)^2 + 4 \\ &= 7(4k^2 + 4k + 1) + 4 \\ &= 28k^2 + 28k + 13 \\ &= 2(14k^2 + 14k + 6) + 1 \end{aligned}$$

As $14k^2 + 14k + 6$ is an integer, $7n^2 + 4$ is odd, and therefore $\neg p$ is true.

We have shown that $\neg q \to \neg p$ is true, therefore $p \to q$ is true.

2) *2) Let a, b, and c be consecutive integers with $a < b < c$. Show that if $a \neq -1$ and $a \neq 3$, then $a^2 + b^2 \neq c^2$.*

We want to prove an implication of the form $p \to q$ is true, with:

$$\begin{array}{cc} p : a \neq -1 \text{ and } a \neq 3 & \neg p : a = -1 \text{ or } a = 3 \\ q : a^2 + b^2 \neq c^2 & \neg q : a^2 + b^2 = c^2 \end{array}$$

Clearly, it is easier to use an indirect proof.

We assume that $\neg q$ is true, i.e. that $a^2 + b^2 = c^2$. Recall that a, b, and c are **consecutive integers** with $a < b < c$. Then:

$$\begin{aligned} b &= a + 1 \\ c &= b + 1 = a + 2 \end{aligned}$$

Therefore, the equation $a^2 + b^2 = c^2$ becomes:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + (a+1)^2 &= (a+2)^2 \\ a^2 + a^2 + 2a + 1 &= a^2 + 4a + 4 \\ a^2 - 2a - 3 &= 0 \\ (a+1)(a-3) &= 0 \end{aligned}$$

whose solutions are $a = -1$ or $a = 3$, i.e. that $\neg p$ is true.

We have shown that $\neg q \to \neg p$ is true, therefore $p \to q$ is true.

3) *Let a and b be two positive real numbers. Use a proof by contradiction to show that if $\frac{a}{b+1} = \frac{b}{a+1}$, then $a = b$.*

We want to prove an implication of the form $p \to q$ is true, with:

$$\begin{array}{cc} p : \frac{a}{b+1} = \frac{b}{a+1} & \neg p : \frac{a}{b+1} \neq \frac{b}{a+1} \\ q : a = b & \neg q : a \neq b \end{array}$$

We use a proof by contradiction:

We assume that p is true, and q is false. Since p is true,

$$\begin{aligned} \frac{a}{b+1} &= \frac{b}{a+1} \\ a(a+1) &= b(b+1) \\ a^2 - b^2 + (a-b) &= 0 \\ (a-b)(a+b+1) &= 0 \end{aligned}$$

As q is false, $a \neq b$ and therefore $a - b \neq 0$. Therefore,

$$a + b = -1$$

However, a and b are both positive: we have reached a contradiction. Therefore $p \to q$ is true.

4) *Prove that $n^2 + n + 9$ is odd for all integer n*

Let n be an integer. We know that $n(n+1) = n^2 + n$ is an even number. Therefore, there exists an integer k such that:

$$n^2 + n = 2k$$

Therefore,

$$n^2 + n + 9 = 2k + 9 = 2(k+4) + 1$$

Since k+4 is an integer, we have that $n^2 + n + 9$ is odd.

5) *Let a and b be two integers. Use a direct proof to show that if $a^2 + 4b^2 - 4ab$ is even, then $a + 2b$ is even.*

We want to prove an implication of the form $p \to q$ is true, with:

$$\begin{array}{cc} p : a^2 + 4b^2 - 4ab \text{ is even} & \neg p : a^2 + b^2 - 4ab \text{ is odd} \\ q : a + 2b \text{ is even} & \neg q : a + 2b \text{ is odd} \end{array}$$

We use a direct proof:

We suppose that p is true: there exists an integer k such that $a^2 + 4b^2 - 4ab = 2k$. Note that

$$\begin{aligned} (a+2b)^2 &= a^2 + 4b^2 + 4ab \\ &= 2k + 4ab + 4ab \\ &= 2(k + 4ab) \end{aligned}$$

Since $k + 2ab$ is an integer, $(a+2b)^2$ is even. Therefore $a + 2b$ is even (using the "Prayer"), i.e. q is true.

We have therefore shown that $p \to q$ is true.