Homework 1 - For 1/12/2022

Exercise 1 (10 points)

Fifty-six biscuits are to be fed to ten pets; each pet is either a cat or a dog. Each dog is to get six biscuits, and each cat is to get five. How many dogs and how many cats are there?

It is best to formalize the problem. Let $C$ be the number of cats, and $D$ the number of dogs. We translate the information we have as relationships on $C$ and $D$:

- There are 10 pets total: $C + D = 10$
- Each dog eats 6 biscuits, each cat eats 5 biscuits, and there are 56 biscuits: $6D + 5C = 56$

The corresponding system of equation

\[
\begin{align*}
C + D &= 10 \\
6D + 5C &= 56
\end{align*}
\]

leads to $D = 6$ and $C = 4$.

Interestingly, there is another way to solve this problem. Each pet eats at least 5 biscuits. As there are 10 pets, this leads to 50 biscuits, and therefore there are only 6 biscuits left. Those biscuits are then given to dogs, one per dog... and therefore there are 6 dogs, and consequently 4 cats.

Exercise 2 (10 points)

A baseball and a baseball bat cost $1.10. The bat costs $1.0 more than the ball. How much does the ball cost?

Intuitively, we would say the bat costs $1.0 and the ball costs $0.10 but we would be wrong!. We need a more systematic approach to solve this problem. Let $x$ be the cost of the ball, and $y$ the cost of the bat. We translate the two sentences given to us as:

\[
\begin{align*}
x + y &= 1.10 \\
y &= x + 1.0
\end{align*}
\]

The solution to this system of equations is $x = $0.05 and $y = $1.05.
Exercise 3 (10 points)

Let $A$ and $B$ be two natural numbers. Follow the proof given below and identify which step(s) is (are) not valid.

<table>
<thead>
<tr>
<th>Step #</th>
<th>Equation</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$A = B$</td>
<td>Assumption</td>
</tr>
<tr>
<td>2</td>
<td>$A \times A = B \times A$</td>
<td>Multiply by $B$ on each side</td>
</tr>
<tr>
<td>3</td>
<td>$A^2 - B^2 = AB - B^2$</td>
<td>Subtract $B^2$ on each side</td>
</tr>
<tr>
<td>4</td>
<td>$(A - B)(A + B) = (A - B)B$</td>
<td>Factorize</td>
</tr>
<tr>
<td>5</td>
<td>$A + B = B$</td>
<td>Simplify: divide by $A-B$</td>
</tr>
<tr>
<td>6</td>
<td>$B + B = B$</td>
<td>Base on step 1, $A = B$, therefore $A + B = B + B$</td>
</tr>
<tr>
<td>7</td>
<td>$2B = B$</td>
<td>By definition, $B + B = 2B$</td>
</tr>
<tr>
<td>8</td>
<td>$2 = 1$</td>
<td>Simplify: divide by $B$</td>
</tr>
</tbody>
</table>

There is only one mistake in the proof, in step 5: we cannot divide by $A - B$ as $A = B$, i.e. $A - B = 0$!!

Exercise 4 (5 points each; total 20 points)

Prove the following identities for $p, q, m, n, x, y$ real numbers:

a) $8(p - q) + 4(p + q) = 2(p + 3q) + 10(p - q)$

Let $p$ and $q$ be two real numbers, and let $LHS = 8(p - q) + 4(p + q)$ and $RHS = 2(p + 3q) + 10(p - q)$. Then:

$$LHS = 8p - 8q + 4p + 4q$$
$$= 12p - 4q$$

and

$$RHS = 2p + 6q + 10p - 10q$$
$$= 12p - 4q$$

Therefore $LHS = RHS$ for all $p$ and $q$, and the identity is true.

b) $x(m - n) + y(n + m) = m(x + y) + n(y - x)$

Let $x, y, m$ and $n$ be four real numbers, and let $LHS = x(m - n) + y(n + m)$ and $RHS = m(x + y) + n(y - x)$. Then:

$$LHS = x m - x n + y n + y m$$

and

$$RHS = x m - x n + y m + y n$$

Therefore $LHS = RHS$ for all $x, y, n$ and $m$, and the identity is true.
c) 

\[(x + 3)(x + 8) - (x - 6)(x - 4) = 21x\]

Let \(x\) be a real number and let \(LHS = (x + 3)(x + 8) - (x - 6)(x - 4)\) and \(RHS = 21x\). Then:

\[
LHS = x^2 + 8x + 3x + 24 - x^2 + 4x + 6x - 24
\]
\[
= 21x
\]
\[
= RHS
\]

The identity is true for all \(x\).

d) 

\[m^8 - 1 = (m^2 - 1)(m^2 + 1)(m^4 + 1)\]

Let \(m\) be a real number and let \(LHS = m^8 - 1\) and \(RHS = (m^2 - 1)(m^2 + 1)(m^4 + 1)\). Then

\[
LHS = \frac{(m^4)^2 - 1^2}{(m^2)^2 - 1^2}
\]
\[
= (m^4 - 1)(m^4 + 1)
\]
\[
= ((m^2)^2 - 1)(m^4 + 1)
\]
\[
= (m^2 - 1)(m^2 + 1)(m^4 + 1)
\]
\[
= RHS
\]

The identity is true for all \(m\).

Exercise 5 (10 points)

A contestant in a TV game show is presented with three boxes, A, B, and C. He is told that one of the boxes contains a prize, while the two others are empty. Each box has a statement written on it:

Box A: The prize is in this box
Box B: The prize is not in box A
Box C: The prize is not in this box

The host of the show tells the contestant that only one of the statements is true. Can the contestant find logically which box contains the prize? Justify your answer.

The simplest approach to solve this problem is to check systematically if A, B, or C contains the prize. In each case, we test the validities of the three statements.

<table>
<thead>
<tr>
<th>Box with prize</th>
<th>Box A</th>
<th>Box B</th>
<th>Box C</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>prize is in this box</td>
<td>prize is not in box A</td>
<td>prize is not in this box</td>
</tr>
<tr>
<td>Box A</td>
<td>True</td>
<td>False</td>
<td>True</td>
</tr>
<tr>
<td>Box B</td>
<td>False</td>
<td>True</td>
<td>True</td>
</tr>
<tr>
<td>Box C</td>
<td>False</td>
<td>True</td>
<td>False</td>
</tr>
</tbody>
</table>

If the prize were in box A or B, two of the propositions would be true, while if the prize is in box C, only one proposition would be true. The latter is therefore true, and the prize is in box C.
Exercise 6 *(10 points)*

This exercise relate to the inhabitants of the island of knights and knaves created by Smullyan, where knights always tell the truth and knaves always lie. You encounter two people, John, Kari, and Tania. John and Kari make the following statements:

*John:* “We are all knaves”
*Kari:* “Exactly one of us is a knight”

What are John, Kari, and Tania?

We proceed as in the discussion section. We check all possible *values* for John, Kari, and Tania, as well as the veracity of their statements:

<table>
<thead>
<tr>
<th>Line number</th>
<th>John</th>
<th>Kari</th>
<th>Tania</th>
<th>John says</th>
<th>Kari says</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Knight</td>
<td>Knight</td>
<td>Knight</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>2</td>
<td>Knight</td>
<td>Knight</td>
<td>Knave</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>3</td>
<td>Knight</td>
<td>Knave</td>
<td>Knight</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>4</td>
<td>Knight</td>
<td>Knave</td>
<td>Knave</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>5</td>
<td>Knave</td>
<td>Knight</td>
<td>Knight</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>6</td>
<td>Knave</td>
<td>Knight</td>
<td>Knave</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>7</td>
<td>Knave</td>
<td>Knave</td>
<td>Knight</td>
<td>F</td>
<td>T</td>
</tr>
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<td>8</td>
<td>Knave</td>
<td>Knave</td>
<td>Knave</td>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>

We can eliminate:

- Line 1, as John would be a knight but he lies
- Line 2, as John would be a knight but he lies
- Line 3, as John would be a knight but he lies
- Line 4, as John would be a knight but he lies
- Line 5, as Kari would be a knight but she lies
- Line 7, as Kari would be a knave but she tells the truth
- Line 8, as John would be a knave but he tells the truth

Line 6 is valid, and it is the only one. Therefore, John is a knave, Kari is a Knight, and Tania is a Knave.