Data, Logic, and Computing

ECS 17 (Winter 2026)

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Homework 6

Exercise 1

Let p, q, and r be three propositions. Using truth tables or logical equivalences, indicate which (if any) of the propositions below are tautologies, contradictions, or neither.

a) $A = (p \wedge q) \vee r \vee (\neg q \wedge \neg r) \vee (\neg p \wedge \neg r)$

Let us do it first using a truth table

p	q	r	$p \wedge q$	$\neg q \wedge \neg r$	$\neg p \wedge \neg r)$	A
T	T	T	T	F	F	T
T	T	F	T	F	F	T
T	F	T	F	F	F	T
T	F	F	F	T	F	T
F	T	T	F	F	F	T
F	T	F	F	F	T	T
F	F	T	F	F	F	T
F	F	F	F	T	T	T

The proposition A is a tautology.

Let us do it now using logical equivalences:

$$\begin{aligned}
A &\Leftrightarrow (p \wedge q) \vee r \vee (\neg q \wedge \neg r) \vee (\neg p \wedge \neg r) \\
&\Leftrightarrow (p \wedge q) \vee r \vee [(\neg q \vee \neg p) \wedge \neg r] \\
&\Leftrightarrow (p \wedge q) \vee r \vee [(\neg(p \wedge q) \wedge \neg r] \\
&\Leftrightarrow (p \wedge q) \vee [(r \vee (\neg(p \wedge q)) \wedge (r \vee \neg r)] \\
&\Leftrightarrow (p \wedge q) \vee [(r \vee (\neg(p \wedge q)) \wedge T] \\
&\Leftrightarrow (p \wedge q) \vee (r \vee (\neg(p \wedge q)) \\
&\Leftrightarrow T \vee r \\
&\Leftrightarrow T
\end{aligned}$$

The proposition A is a tautology.

b) $[p \vee (q \rightarrow r)] \rightarrow (p \vee q \vee r)$

Let us define $A = p \vee (q \rightarrow r)$ and $B = p \vee q \vee r$. Let us build the truth table:

p	q	r	$q \rightarrow r$	A	B	$A \rightarrow B$
T	T	T	T	T	T	T
T	T	F	F	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	T	T
F	T	T	T	T	T	T
F	T	F	F	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	F	F

The proposition $A \rightarrow B$ is neither a tautology nor a contradiction

Exercise 2

Prove that if n is a positive integer, then n is even if and only if $7n + 4$ is even.

Let p be the proposition "n is even" and q be the proposition "$7n$+4 is even". We want to show that $p \leftrightarrow q$ is true, which is logically equivalent to show that $p \rightarrow q$ and $q \rightarrow p$.

i) Let us show $p \rightarrow q$:

Hypothesis: p is true, i.e. n is even. If n is even, then there exists an integer k such that let $n = 2k$,We get:

$$\begin{aligned}
7n + 4 &= 7(2k) + 4 \\
&= 14k + 4 \\
&= 2(7k + 2)
\end{aligned}$$

As $7k+2$ is an integer, $7n+4$ is a multiple of 2: it is even.

ii) Let us show $q \rightarrow p$:

Hypothesis: q is true, i.e. $7n$+4 is even. If $7n+4$ is even, then there exists an integer k such $7n+4 = 2k$. We get:

$$\begin{aligned} 7n &= 2k-4 \\ n &= 2k-4-6n \\ n &= 2*(k-2-3n) \end{aligned}$$

As $k-2-3n$ is an integer, n is a multiple of 2: it is even. Note that we could have done this proof using a contrapositive.

We conclude: "n is even" and " $7n$+4 is even" are logically equivalent.

Exercise 3

Let a and b be two positive integers. Prove that if $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$

Let p be the proposition "$n = ab$", and let q be the proposition $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$. We use a proof by contradiction, namely we assume that p is true AND q is false.

Since q is false, we know that

$$\begin{aligned} a &> \sqrt{n} \quad (1) \\ b &> \sqrt{n}. \quad (2) \end{aligned}$$

As $\sqrt{n}$ is positive, we have that both $a > 0$ and $b > 0$. We multiply (1) with b and (2) with $\sqrt{n}$:

$$\begin{aligned} ab &> b\sqrt{n} \quad (3) \\ b\sqrt{n} &> n. \quad (4) \end{aligned}$$

By transitivity, we get that $ab > n$, but we also have $ab = n$ as p is true. We have reached a contradiction. Therefore the original implication is true.

Exercise 4

Let m and n be two integers. Show that if $m > 0$ and $n \leq -2$, then $m^2 + mn + n^2 \geq 0$

We use a direct proof. Let p be the proposition $m > 0$ and $n \leq -2$, and let q be the proposition $m^2 + mn + n^2 \geq 0$. To show $p \rightarrow q$, we will show that if p is true, then q is also true.

Let m and n be two integers.

Hypothesis: p is true. Therefore $m > 0$ and $n \leq -2$. As m is (strictly) positive, we can multiply $n \leq -2$ by m without changing the sense of the inequality; then $mn \leq -2m$. As m is strictly positive, $-2m$ is strictly negative. Therefore:

$$\begin{aligned} mn &\leq -2m \\ -2m &< 0 \end{aligned}$$

i.e. $mn < 0$.

Let us consider now:

$$\begin{aligned} m^2 + mn + n^2 &= m^2 + 2mn + n^2 - mn \\ &= (m+n)^2 - mn \end{aligned}$$

Note that $(m+n)^2$ is positive and $-mn$ is also positive, as mn is negative. $m^2 + mn + n^2$ is the sum of two positive numbers; it is positive. Therefore q is true, and the property is true.

Exercise 5

Let a and b be two integers. Show that if $a^2 + b^2$ is even, then $a + b$ is even:

We define:
p: $a^2 + b^2$ is even
q: $a + b$ is even

a) *Using an indirect proof (proof by contrapositive)*

Hypothesis: $a + b$ is odd. Therefore, there exists an integer k such that $a + b = 2k + 1$. We compute $(a+b)^2$ in two different ways,

$$\begin{aligned} (a+b)^2 &= a^2 + b^2 + 2ab \\ (a+b)^2 &= 4k^2 + 4k + 1 \end{aligned}$$

. Therefore,

$$\begin{aligned} a^2 + b^2 &= 4k^2 + 4k + 1 - 2ab \\ &= 2(2k^2 + 2k - ab) + 1 \end{aligned}$$

As $2k^2 + 2k - ab$ is an integer, we get that $a^2 + b^2$ is odd, i.e. $\neg p$ is true. This concludes the indirect proof.

b) *Using a proof by contradiction*

Hypothesis: p is true AND $\neg q$ is true. Therefore, we know that $a^2 + b^2$ is even, and $a + b$ is odd. As $a + b$ is odd, there exists an integer k such that $a + b = 2k + 1$. We compute $(a+b)^2$ in two different ways,

$$\begin{aligned} (a+b)^2 &= a^2 + b^2 + 2ab \\ (a+b)^2 &= 4k^2 + 4k + 1 \end{aligned}$$

. Therefore,

$$\begin{aligned} a^2 + b^2 &= 4k^2 + 4k + 1 - 2ab \\ &= 2(2k^2 + 2k - ab) + 1 \end{aligned}$$

As $2k^2 + 2k - ab$ is an integer, we get that $a^2 + b^2$ is odd, i.e. $\neg p$ is true. However, we had assumed that p is true: we have reached a contradiction. This concludes the proof by contradiction.

c) *Using a direct proof*

Hypothesis: p is true. Then a^2+b^2 is even, i.e. there exists an integer k such that $a^2+b^2=2k$. Note that

$$\begin{aligned}(a+b)^2 &= a^2+b^2+2ab\\ &= 2k+2ab\\ &= 2(k+ab)\end{aligned}$$

As $k+ab$ is an integer, $(a+b)^2$ is even, and therefore $a+b$ is even. This concludes the direct proof.

Exercise 6

This exercise relates to the inhabitants of the island of knights and knaves, where knights always tell the truth and knaves always lie. John and Bill are residents. John tells you: "if Bill is a knave, then I am a knight", while Bill tells you: "if John is a knave and I am a knight, then 2+2=5". Can you say what John and Bill are?

We proceed as usual: we check all possible "values" for John and Bill, as well as the veracity of their statements. Note that both statements are implications.

Line number	John	Bill	John says	Bill says
1	Knight	Knight	T	T
2	Knight	Knave	T	T
3	Knave	Knight	T	F
4	Knave	Knave	F	T

We can eliminate:

- Line 2, as Bill would be a knave but he tells the truth
- Line 3 as John would be a knave but he says the true
- Line 4, as Bill would be a knave but he tells the truth

Line 1 is valid, and it is the only one. Therefore, both John and Bill are knights.