Data, Logic, and Computing

ECS 17 (Winter 2025)

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Homework 8

Exercise 1

Show that $\forall n \in \mathbb{N}, \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$

Let P(n) be the proposition: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{2}$. Let us also define $LHS(n) = \sum_{i=1}^{n} i^2$ and $RHS(n) = \frac{n(n+1)(2n+1)}{2}$

• Basis step: P(1) is true:

$$LHS(1) = \sum_{i=1}^{1} i^2 = 1$$

RHS(1) = $\frac{1(1+1)(2+1)}{6} = \frac{2 \times 3}{6} = 1$

• Inductive step: Let k be a positive integer $(k \le 0)$, and let us suppose that P(k) is true. We want to show that P(k+1) is true.

Let us compute $LHS(k+1) = \sum_{i=1}^{k+1} i^2$:

$$LHS(k+1) = \sum_{i=1}^{k} i^{2} + (k+1)^{2}$$

= $LHS(k) + (k+1)^{2}$
= $RHS(k) + (k+1)^{2}$
= $\frac{k(k+1)(2k+1)}{6} + (k+1)^{2}$
= $\frac{k(k+1)(2k+1) + 6(k+1)^{2}}{6}$
= $\frac{(k+1)(2k^{2} + k + 6k + 6)}{6}$
= $\frac{(k+1)(2k^{2} + 7k + 6)}{6}$
= $\frac{(k+1)(k+2)(2k+3)}{6}$

And:

$$RHS(k+1) = \frac{(k+1)(k+2)(2k+3)}{6}$$

Therefore LHS(k+1) = RHS(k+1), which validates that P(k+1) is true.

The principle of proof by mathematical induction allows us to conclude that P(n) is true for all n.

Exercise 2

Show that
$$\forall n \in \mathbb{N}, \sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}.$$

Let
$$P(n)$$
 be the proposition: $\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}$. We define $LHS(n) = \sum_{i=1}^{n} i(i+1)(i+2)$ and $RHS(n) = \frac{n(n+1)(n+2)(n+3)}{4}$

• Basis step: P(1) is true:

$$LHS(1) = 1 * (1+1) * (1+2) = 6$$

RHS(1) = $\frac{1 * (1+1) * (1+2) * (1+3)}{4} = 6$

• Inductive step: Let k be a positive integer $(k \le 0)$, and let us suppose that P(k) is true. We want to show that P(k+1) is true.

Let us compute LHS(k+1):

$$LHS(k+1) = \sum_{i=1}^{k+1} i(i+1)(i+2)$$

= $LHS(k) + (k+1)(k+2)(k+3)$
= $\frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)$
= $\frac{k(k+1)(k+2)(k+3)}{4} + \frac{4(k+1)(k+2)(k+3)}{4}$
= $\frac{(k+1)(k+2)(k+3)(k+4)}{4}$

Let us compute RHS(k+1):

$$RHS(k+1) = \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

Therefore LHS(k+1) = RHS(k+1), which validates that P(k+1) is true.

The principle of proof by mathematical induction allows us to conclude that P(n) is true for all n.

Exercise 3

Show that $\forall n \in \mathbb{N}, n > 1, \sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}.$

Let P(n) be the proposition: $\sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}$. Let us define $LHS(n) = \sum_{i=1}^{n} \frac{1}{i^2}$ and $RHS(n) = 2 - \frac{1}{n}$. We want to show that P(n) is true for all n > 1.

• Basis step: We show that P(2) is true:

$$LHS(2) = 1 + \frac{1}{4} = \frac{5}{4}$$
$$RHS(2) = 2 - \frac{1}{2} = \frac{6}{4}$$

Therefore LHS(2) < RHS(2) and P(2) is true.

• Inductive step: Let k be a positive integer greater than 1 (k > 1), and let us suppose that P(k) is true. We want to show that P(k+1) is true.

$$LHS(k+1) = LHS(k) + \frac{1}{(k+1)^2}$$

Since P(k) is true, we find:

 $LHS(k+1) < 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$ Since k + 1 > k, $\frac{1}{(k+1)^2} < \frac{1}{k(k+1)}$. Therefore $LHS(k+1) < 2 - \frac{1}{k} + \frac{1}{k(k+1)}$ We can use the property : $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$: $LHS(k+1) < 2 - \frac{1}{k} + \frac{1}{k} - \frac{1}{k+1}$ $LHS(k+1) < 2 - \frac{1}{k+1}$ Since $RHS(k+1) = 2 - \frac{1}{k+1}$, we get LHS(k+1) < RHS(k+1) which validates that P(k+1) is true.

The principle of proof by mathematical induction allows us to conclude that P(n) is true for all n > 1.

Exercise 4

Use a proof by induction to show that $\forall n \in \mathbb{N}, n > 3, n^2 - 7n + 12 \ge 0$.

Let P(n) be the proposition: $n^2 - 7n + 12 \ge 0$. We want to show that P(n) is true for n greater than 3. Let us define $LHS(n) = n^2 - 7n + 12$. Notice that LHS(1) = 6, LHS(2) = 2 and LHS(3) = 0 hence P(1), P(2) and P(3) are true.

• Basis step: P(4) is true:

$$LHS(4) = 4^2 - 7 * 4 + 12 = 0$$

Therefore $LHS(4) \ge 0$ and P(4) is true.

• Inductive step: Let k be a positive integer greater than 3 (k > 3), and let us suppose that P(k) is true. We want to show that P(k+1) is true.

$$LHS(k+1) = (k+1)^2 - 7(k+1) + 12$$

= $k^2 + 2k + 1 - 7k - 7 + 12$
= $(k^2 - 7k + 12) + (2k - 6)$

Since P(k) is true, we know that $k^2 - 7k + 12 \ge 0$. Since $k \ge 4$, 2k - 6 > 0. Therefore, $(k+1)^2 - 7(k+1) + 12 > 0.$

This validates that P(k+1) is true.

The principle of proof by mathematical induction allows us to conclude that P(n) is true for all n > 3.

Exercise 5

A sequence a_0, a_1, \ldots, a_n of natural numbers is defined by $a_0 = 2$ and $a_{n+1} = (a_n)^2$, $\forall n \in \mathbb{N}$. Find a closed form formula for the term a_n and prove that your formula is correct.

Let is first compute a few terms in the sequence:

$$a_0 = 2 = 2^0$$

$$a_1 = (a_0)^2 = 4 = 2^2$$

$$a_2 = (a_1)^2 = 16 = 2^4$$

$$a_3 = (a_2)^2 = 196 = 2^8$$

We notice two things:

- i) each term a_n is a power of 2
- ii) the power coefficient is itself a power of 2

Based on these observations, we assume that $a_n = 2^{2^n}$. Note that this is true for n = 0, n = 1, n = 2, and n = 3. Let us show that it is true for all n non negative integers.

Let us define: $A(n) = 2^{2^n}$ and let us define $P(n) : a_n = A(n)$; we want to show that P(n) is true, for all $n \in \mathbb{Z}, n \ge 0$.

a) Basis step: we want to show that P(0) is true. $a_0 = 2$

 $A(0) = 2^{2^0} = 2^1 = 2$ Therefore $a_0 = A(0)$ and p(0) is true.

b) Inductive Step I want to show $p(k) \rightarrow p(k+1)$ whenever $k \ge 0$

Hypothesis: p(k) is true, i.e. $a_k = A(k)$; i.e. $a_k = 2^{2^k}$. Then:

$$a_{k+1} = (a_k)^2$$
$$= (2^{2^k})^2$$
$$= 2^{2^k \times 2}$$
$$= 2^{2^{k+1}}$$
$$= A_{k+1}$$

Therefore $a_{k+1} = A(k+1)$ which validates that p(k+1) is true.

The principle of proof by mathematical induction allows us to conclude that p(n) is true for all $n \ge 0$.

Exercise 6

Use the method of proof by induction to show that any amount of postage of 24 cents or more can be formed using just 5-cent and 7-cent stamps.

Let P(n) be the property: the amount of postage of n cents can be formed using just 5-cent and 7-cent stamps. We want the show that P(n) is true, for all $n \ge 24$.

Let us first analyze what this property means. We can rewrite it as: "There exists two nonnegative integers m and p such that n = 5m + 7p. We prove the property using induction.

• Basis step: We want to show that P(24) is true.

Note that $24 = 5 \times 2 + 7 \times 2$. We found a pair of non negative integers (m, p) = (2, 2) such that 24 = 5m + 7p. P(24) is therefore true.

• *induction step*: We suppose that P(k) is true, for $k \ge 24$, and we want to show that P(k+1) is true.

Since P(k) is true, there exists two non negative integers (m, p) such that

$$k = 5m + 7p$$

Adding 1 to this equation, we get:

$$k+1 = 5m + 7p + 1$$

We notice that 1 can be written as 5 - 4. In which case:

$$k+1 = 5m+7p+15-14 = 5(m+3)+7(p-2)$$

p-1 may not be non-negative however, based on the value of p. We therefore distinguish three cases:

 $-p \ge 2$ In this case, p-2 is non negative. We found a pair of non negative integers (m', p') = (m+3, p-2) such that k+1 = 5m' + 7p'. P(k+1) is therefore true.

-p = 1 In this case, p - 2 is negative. Let us go back to

$$k+1 = 5m+7p+1$$
$$= 5m+8$$

Since p = 1. We note first that $m \ge 4$ as $k \ge 24$. We notice then that 1 = 21 - 20. In this case:

$$k+1 = 5m+28-20 = 5(m-4) + 7 \times 4$$

with 4 and m-4 being non negative. We found a pair of non negative integers (m', p') = (m-4, 4) such that k+1 = 5m' + 7p'. P(k+1) is therefore true.

 $-\ p=0$ In this case, p-2 is also negative. Let us go back to

$$k+1 = 5m+7p+1$$
$$= 5m+1$$

Since p = 0. We note again that $m \ge 4$ as $k \ge 24$. We notice then that 1 = 21 - 20. In this case:

$$k+1 = 5m+21-20 = 5(m-4)+7 \times 3$$

with 3 and m-4 being non negative. We found a pair of non negative integers (m', p') = (m-4, 3) such that k+1 = 5m' + 7p'. P(k+1) is therefore true.

In all cases, P(k+1) is true.

The principle of proof by mathematical induction allows us to conclude that P(n) is true for all $n \ge 24$.