Homework 8 - For 3/7/2022

Exercise 1

Show that \( \forall n \in \mathbb{N} f_1^2 + f_2^2 + \ldots + f_n^2 = f_n f_{n+1} \) where \( f_n \) are the Fibonacci numbers.

Let \( P(n) \) be the proposition: \( f_1^2 + f_2^2 + \ldots + f_n^2 = f_n f_{n+1} \)

where \( f_n \) are the Fibonacci numbers. Let us define \( LHS(n) = f_1^2 + f_2^2 + \ldots + f_n^2 \) and \( RHS(n) = f_n f_{n+1} \).

We want to show that \( P(n) \) is true for all \( n \); we use a proof by induction.

- **Basis step**: \( P(1) \) is true:
  \[
  LHS(2) = f_1^2 = 1^2 = 1 \\
  RHS(2) = f_1 f_2 = 1.
  \]

- **Inductive step**: Let \( k \) be a positive integer, and let us suppose that \( P(k) \) is true. We want to show that \( P(k + 1) \) is true.

  Then
  \[
  LHS(k + 1) = f_1^2 + f_2^2 + \ldots + f_k^2 + f_{k+1}^2 \\
  = f_k f_{k+1} + f_{k+1}^2 \\
  = f_{k+1} (f_k + f_{k+1}) \\
  = f_{k+1} f_{k+2}
  \]

  and
  \[
  RHS(k + 1) = f_{k+1} f_{k+2}
  \]

  Therefore \( LHS(k + 1) = RHS(k + 1) \), which validates that \( P(k + 1) \) is true.

The principle of proof by mathematical induction allows us to conclude that \( P(n) \) is true for all \( n \).
Exercise 2

Show that $\forall n \in \mathbb{N} f_0 - f_1 + f_2 \ldots - f_{2n-1} + f_{2n} = f_{2n-1} - 1$ where $f_n$ are the Fibonacci numbers.

Let $P(n)$ be the proposition: $f_0 - f_1 + f_2 \ldots - f_{2n-1} + f_{2n} = f_{2n-1} - 1$

where $f_n$ are the Fibonacci numbers. Let us define $LHS(n) = f_0 - f_1 + f_2 \ldots - f_{2n-1} + f_{2n}$ and $RHS(n) = f_{2n-1} - 1$.

We want to show that $P(n)$ is true for all $n > 0$; we use a proof by induction.

- **Basis step:**
  
  $$LHS(1) = f_0 - f_1 + f_2 = 0 - 1 + 1 = 0$$
  $$RHS(1) = f_1 - 1 = 1 - 1 = 0$$

  Therefore $LHS(1) = RHS(1)$ and $P(1)$ is true.

- **Inductive step:** Let $k$ be a positive integer, and let us suppose that $P(k)$ is true. We want to show that $P(k+1)$ is true.

  Then

  $$LHS(k+1) = f_0 - f_1 + \ldots - f_{2k-1} + f_{2k} = f_{2k-1} - 1 - f_{2k+1} + f_{2k+2}$$
  
  $$= f_{2k-1} - 1 - f_{2k+1} + f_{2k+2}$$
  
  $$= f_{2k-1} - 1 - f_{2k+1} + (f_{2k} + f_{2k+1})$$
  
  $$= f_{2k-1} + f_{2k} - 1$$
  
  $$= f_{2k+1} - 1$$

  and

  $$RHS(k+1) = f_{2k+1} - 1$$

  Therefore $LHS(k+1) = RHS(k+1)$, which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n$.

Exercise 3

a) **How many strings of four lowercase letters have the letter x in them?**

  There are 26 lowercase letters. Consequently, there are $26^4$ 4-letter strings using all the 26 letters, and there are $25^4$ 4-letter strings using all letters except “x”. Thus, using the rule of complement, the number of strings that have the letter “x” in them is $26^4 - 25^4 = 66351$.

b) **How many strings of four decimal digits do not contain the same digit twice?**

To build a string with 4 distinct digits, we have 10 choices for the first digit, 9 choices for the second digit, 8 choices for the third digit, and 7 choices for the fourth digit. Based on the product rule, we conclude that there are $10 \times 9 \times 8 \times 7 = 5040$ four decimal digits that do not contain the same digit twice.
c) How many strings of four decimal digits end with an even digit?

Strings that end with an even digit have 10 possibilities (i.e., from \{0, 1, 2, \ldots, 9\}) for the first 3 digits and 5 possibilities (i.e., from \{0, 2, 4, \ldots, 6, 8\}) for the last digit. By the product rule, there are \(10 \times 10 \times 10 \times 5 = 5000\) strings which end with an even digit.

d) How many strings of four decimal digits contain exactly three digits that are 9s?

The 4-decimal string which have exactly 3 digits that are 9’s has one non-9 digit. Thus, there are 9 possibilities for the 4th digit (i.e., from \{0, 1, 2, \ldots, 8\}). The non-digit can be in position 1, 2, 3 or 4 in the string. Based on the sum rule, we conclude that there are \(9 + 9 + 9 + 9 = 36\) four decimal digits that have exactly three 9.

Exercise 4

How many bit strings of length 10 contain either exactly five consecutive 0s or exactly five consecutive 1s?

Let \(A\) be the set of bit strings of length 10 which contain five consecutive 0’s. We treat the five 0’s as a block \(p\); assuming that there is exactly five consecutive 0’s, we have to place 1 on its sides.

- If \(p\) starts in position 1 (left most bit), the first 5 bits of the string are 0, and the sixth string is 1. Each of the last four bits can be chosen in 0,1, and therefore there are \(2^4\) such strings.
- If \(p\) starts in position 2, the first bit is 1, then bits 2-6 are 0, and bit 7 is 1. Bits 8, 9 and 10 are each chosen in 0,1, and therefore there are \(2^3\) such strings.
- If \(p\) starts in position 3, the first bit can be 0 or 1, then bit 2 is 1, bits 3-7 are 0 and bit 8 is 1. Bits 9 and 10 are each chosen in 0,1. There are therefore \(2^3\) such strings.
- If \(p\) starts in position 4, each of the first two bits is chosen in 0,1, then bit 3 is 1, bits 4-8 are 0 and bit 9 is 1. Bit 10 is chosen in 0,1. There are \(2^3\) such strings.
- If \(p\) starts in position 5, the each of the first three bits is chosen in 0,1, then bit 4 is 1, bits 5-9 are 0, and bit 10 is 1. There are \(2^3\) such strings.
- If \(p\) starts in position 6, then each of the first 4 bits is chosen in 0,1, bit 5 is 1, and bits 6-10 are 0. There are \(2^4\) such strings.

As these 5 cases are mutually exclusive, we apply the sum rule, and find that there are \(2 \times 2^4 + 4 \times 2^3 = 64\) strings with exactly 5 consecutive 0s.

Using the same reasoning, there are 64 strings with exactly 5 consecutive 1s.

There are two strings that have both 5 consecutive 0 and 5 consecutive 1: 111100000 and 0000011111. Using the inclusion-exclusion principle (i.e. generalized sum rule), the number of strings that contain either five consecutive 0’s or five consecutive 1’s is \(64 + 64 - 2 = 126\).

Exercise 5

Use a tree diagram to find the number of bit strings of length four with no three consecutive 0s.

As shown in Figure 1, from the tree diagram we get that there can be 13 4-bit strings with no three consecutive 0’s.
Exercise 6

From a group of 13 men, 8 women, 2 boys and 4 girls,

(a) How many ways can a man, a woman, a boy and a girl be selected?
Using the product rule, there are $13 \times 8 \times 2 \times 4 = 832$ ways of selecting a man, a woman, a boy and a girl from 13 men, 8 women, 2 boys and 4 girls.

(b) How many ways can a male and a female be selected?
There are $13 + 2 = 15$ males and $8 + 4 = 12$ females. Using the product rule, there are $15 \times 12 = 180$ ways of selecting a male and a female.

(c) How many ways can a person be selected?
There are $15 + 12 = 27$ people in the group. Hence, there are 27 ways of selecting one person.