Data, Logic, and Computing

ECS 17 (Winter 2024)

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Homework 8 - For 3/06/2024

Exercise 1

Show that $\forall n \in \mathbb{N}, \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Let $P(n)$ be the proposition: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{2}$. Let us also define $LHS(n) = \sum_{i=1}^{n} i^2$ and $RHS(n) = \frac{n(n+1)(2n+1)}{2}$

- *Basis step:* $P(1)$ is true:

$$\begin{aligned} LHS(1) &= \sum_{i=1}^{1} i^2 = 1 \\ RHS(1) &= \frac{1(1+1)(2+1)}{6} = \frac{2 \times 3}{6} = 1 \end{aligned}$$

- *Inductive step:* Let k be a positive integer ($k \leq 0$), and let us suppose that $P(k)$ is true. We want to show that $P(k+1)$ is true.

  Let us compute $LHS(k+1) = \sum_{i=1}^{k+1} i^2$:

$$
\begin{aligned}
LHS(k+1) &= \sum_{i=1}^{k} i^2 + (k+1)^2 \\
&= LHS(k) + (k+1)^2 \\
&= RHS(k) + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}
$$

And:

$$RHS(k+1) = \frac{(k+1)(k+2)(2k+3)}{6}$$

Therefore $LHS(k+1) = RHS(k+1)$, which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all n.

Exercise 2

Show that $\forall n \in \mathbb{N}, \sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}$.

Let $P(n)$ be the proposition: $\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}$. We define $LHS(n) = \sum_{i=1}^{n} i(i+1)(i+2)$ and $RHS(n) = \frac{n(n+1)(n+2)(n+3)}{4}$

- *Basis step:* $P(1)$ is true:

$$
\begin{aligned}
LHS(1) &= 1 * (1+1) * (1+2) = 6 \\
RHS(1) &= \frac{1 * (1+1) * (1+2) * (1+3)}{4} = 6
\end{aligned}
$$

- *Inductive step:* Let k be a positive integer ($k \leq 0$), and let us suppose that $P(k)$ is true. We want to show that $P(k+1)$ is true.

Let us compute $LHS(k+1)$:

$$\begin{aligned} LHS(k+1) &= = \sum_{i=1}^{k+1} i(i+1)(i+2) \\ &= LHS(k) + (k+1)(k+2)(k+3) \\ &= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) \\ &= \frac{k(k+1)(k+2)(k+3)}{4} + \frac{4(k+1)(k+2)(k+3)}{4} \\ &= \frac{(k+1)(k+2)(k+3)(k+4)}{4} \end{aligned}$$

Let us compute $RHS(k+1)$:

$$RHS(k+1) = \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

Therefore $LHS(k+1) = RHS(k+1)$, which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all n.

Exercise 3

Show that $\forall n \in \mathbb{N}, n > 1, \sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}$.

Let $P(n)$ be the proposition: $\sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}$. Let us define $LHS(n) = \sum_{i=1}^{n} \frac{1}{i^2}$ and $RHS(n) = 2 - \frac{1}{n}$. We want to show that $P(n)$ is true for all $n > 1$.

- *Basis step:* We show that $P(2)$ is true:

$$LHS(2) = 1 + \frac{1}{4} = \frac{5}{4}$$
$$RHS(2) = 2 - \frac{1}{2} = \frac{6}{4}$$

  Therefore $LHS(2) < RHS(2)$ and $P(2)$ is true.

- *Inductive step:* Let k be a positive integer greater than 1 ($k > 1$), and let us suppose that $P(k)$ is true. We want to show that $P(k+1)$ is true.

$$LHS(k+1) = LHS(k) + \frac{1}{(k+1)^2}$$

Since P(k) is true, we find:

$$LHS(k+1) < 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

Since $k+1 > k$, $\frac{1}{(k+1)^2} < \frac{1}{k(k+1)}$.
Therefore

$$LHS(k+1) < 2 - \frac{1}{k} + \frac{1}{k(k+1)}$$

We can use the property : $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$:

$$\begin{aligned} LHS(k+1) &< 2 - \frac{1}{k} + \frac{1}{k} - \frac{1}{k+1} \\ LHS(k+1) &< 2 - \frac{1}{k+1} \end{aligned}$$

Since $RHS(k+1) = 2 - \frac{1}{k+1}$, we get $LHS(k+1) < RHS(k+1)$ which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n > 1$.

Exercise 4

Use a proof by induction to show that $\forall n \in \mathbb{N}, n > 3, n^2 - 7n + 12 \geq 0$.

Let $P(n)$ be the proposition: $n^2 - 7n + 12 \geq 0$. We want to show that $P(n)$ is true for n greater than 3. Let us define $LHS(n) = n^2 - 7n + 12$.
Notice that $LHS(1) = 6$, $LHS(2) = 2$ and $LHS(3) = 0$ hence $P(1)$, $P(2)$ and $P(3)$ are true.

- *Basis step:* $P(4)$ is true:

  $$LHS(4) = 4^2 - 7 * 4 + 12 = 0$$

  Therefore $LHS(4) \geq 0$ and $P(4)$ is true.

- *Inductive step:* Let k be a positive integer greater than 3 ($k > 3$), and let us suppose that $P(k)$ is true. We want to show that $P(k+1)$ is true.

  $$\begin{aligned} LHS(k+1) &= (k+1)^2 - 7(k+1) + 12 \\ &= k^2 + 2k + 1 - 7k - 7 + 12 \\ &= (k^2 - 7k + 12) + (2k - 6) \end{aligned}$$

  Since $P(k)$ is true, we know that $k^2 - 7k + 12 \geq 0$. Since $k \geq 4$, $2k - 6 > 0$. Therefore, $(k+1)^2 - 7(k+1) + 12 > 0$.
  This validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n > 3$.

Exercise 5: 10 points

A sequence $a_0, a_1, \ldots, a_n$ of natural numbers is defined by $a_0 = 2$ and $a_{n+1} = (a_n)^2$, $\forall n \in \mathbb{N}$. Find a closed form formula for the term a_n and prove that your formula is correct.

Let is first compute a few terms in the sequence:

$$\begin{aligned} a_0 &= 2 = 2^0 \\ a_1 &= (a_0)^2 = 4 = 2^2 \\ a_2 &= (a_1)^2 = 16 = 2^4 \\ a_3 &= (a_2)^2 = 196 = 2^8 \end{aligned}$$

We notice two things:

i) each term a_n is a power of 2

ii) the power coefficient is itself a power of 2

Based on these observations, we assume that $a_n = 2^{2^n}$. Note that this is true for $n = 0$, $n = 1$, $n = 2$, and $n = 3$. Let us show that it is true for all n non negative integers.

Let us define: $A(n) = 2^{2^n}$ and let us define $P(n) : a_n = A(n)$; we want to show that $P(n)$ is true, for all $n \in \mathbb{Z}, n \geq 0$.

a) Basis step: we want to show that $P(0)$ is true.
$a_0 = 2$
$A(0) = 2^{2^0} = 2^1 = 2$
Therefore $a_0 = A(0)$ and $p(0)$ is true.

b) Inductive Step
I want to show $p(k) \to p(k+1)$ whenever $k \geq 0$

Hypothesis: $p(k)$ is true, i.e. $a_k = A(k)$¡ i.e. $a_k = 2^{2^k}$.
Then:

$$\begin{aligned} a_{k+1} &= (a_k)^2 \\ &= \left(2^{2^k}\right)^2 \\ &= 2^{2^k \times 2} \\ &= 2^{2^{k+1}} \\ &= A_{k+1} \end{aligned}$$

Therefore $a_{k+1} = A(k+1)$ which validates that $p(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $p(n)$ is true for all $n \geq 0$.

Exercise 6

Show that $\forall n \in \mathbb{N} f_1^2 + f_2^2 + \ldots + f_n^2 = f_n f_{n+1}$ where f_n are the Fibonacci numbers.

Let $P(n)$ be the proposition: $f_1^2 + f_2^2 + \ldots + f_n^2 = f_n f_{n+1}$ where f_n are the Fibonacci numbers. Let us define $LHS(n) = f_1^2 + f_2^2 + \ldots + f_n^2$ and $RHS(n) = f_n f_{n+1}$.

We want to show that $P(n)$ is true for all n; we use a proof by induction.

- *Basis step:* $P(1)$ is true:

$$\begin{aligned} LHS(2) &= f_1^2 = 1^2 = 1 \\ RHS(2) &= f_1 f_2 = 1. \end{aligned}$$

- *Inductive step:* Let k be a positive integer, and let us suppose that $P(k)$ is true. We want to show that $P(k+1)$ is true.
  Then

$$\begin{aligned} LHS(k+1) &= f_1^2 + f_2^2 + \ldots + f_k^2 + f_{k+1}^2 \\ &= f_k f_{k+1} + f_{k+1}^2 \\ &= f_{k+1}(f_k + f_{k+1}) \\ &= f_{k+1} f_{k+2} \end{aligned}$$

  and

$$RHS(k+1) = f_{k+1} f_{k+2}$$

  Therefore $LHS(k+1) = RHS(k+1)$, which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all n.

Exercise 7

Show that $\forall n \in \mathbb{N} f_0 - f_1 + f_2 - \ldots - f_{2n-1} + f_{2n} = f_{2n-1} - 1$ where f_n are the Fibonacci numbers.

Let $P(n)$ be the proposition: $f_0 - f_1 + f_2 - \ldots - f_{2n-1} + f_{2n} = f_{2n-1} - 1$ where f_n are the Fibonacci numbers. Let us define $LHS(n) = f_0 - f_1 + f_2 - \ldots - f_{2n-1} + f_{2n}$ and $RHS(n) = f_{2n-1} - 1$.

We want to show that $P(n)$ is true for all $n > 0$; we use a proof by induction.

- *Basis step:*

$$\begin{aligned} LHS(1) = f_0 - f_1 + f_2 &= 0 - 1 + 1 = 0 \\ RHS(1) = f_1 - 1 &= 1 - 1 = 0 \end{aligned}$$

  Therefore $LHS(1) = RHS(1)$ and $P(1)$ is true.

- *Inductive step:* Let k be a positive integer, and let us suppose that $P(k)$ is true. We want to show that $P(k+1)$ is true.
  Then

$$\begin{aligned} LHS(k+1) &= f_0 - f_1 + ... - f_{2k-1} + f_{2k} - f_{2k+1} + f_{2k+2} \\ &= f_{2k-1} - 1 - f_{2k+1} + f_{2k+2} \\ &= f_{2k-1} - 1 - f_{2k+1} + (f_{2k} + f_{2k+1}) \\ &= f_{2k-1} + f_{2k} - 1 \\ &= f_{2k+1} - 1 \end{aligned}$$

  and

$$RHS(k+1) = f_{2k+1} - 1$$

  Therefore $LHS(k+1) = RHS(k+1)$, which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all n.

Exercise 8: 10 points

Use the method of proof by induction to show that any amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Let $P(n)$ be the property: the amount of postage of n cents can be formed using just 4-cent and 5-cent stamps. We want the show that $P(n)$ is true, for all $n \geq 12$.

Let us first analyze what this property means. We can rewrite it as: "There exists two non-negative integers m and p such that $n = 4m + 5p$. We prove the property using induction.

- *Basis step*: We want to show that $P(12)$ is true.

  Note that $12 = 4 \times 3 + 5 \times 0$. We found a pair of non negative integers $(m, p) = (3, 0)$ such that $12 = 4m + 5p$. P(12) is therefore true.

- *induction step*: We suppose that $P(k)$ is true, for $k \geq 12$, and we want to show that $P(k+1)$ is true.

  Since $P(k)$ is true, there exists two non negative integers (m, p) such that

$$k = 4m + 5p$$

  Adding 1 to this equation, we get:

$$k + 1 = 4m + 5p + 1$$

  We notice that 1 can be written as 5 - 4. In which case:

$$\begin{aligned} k+1 &= 4m + 5p + 5 - 4 \\ &= 4(m-1) + 5(p+1) \end{aligned}$$

  $m - 1$ may not be non-negative however, based on the value of m. We therefore distinguish two cases:

- $m \neq 0$ In this case, $m-1$ is non negative. We found a pair of non negative integers $(m', p') = (m-1, p+1)$ such that $k+1 = 4m' + 5p'$. P(k+1) is therefore true.
- $m = 0$ In this case, $m-1$ is negative. Let us go back to

$$\begin{aligned} k+1 &= 4m + 5p + 1 \\ &= 5p + 1 \end{aligned}$$

  Since $m = 0$. We note first that $p \geq 3$ as $k \geq 12$. We notice then that $1 = 16 - 15$. In this case:

$$\begin{aligned} k+1 &= 5p + 16 - 15 \\ &= 4 \times 4 + 5(p-3) \end{aligned}$$

  with 4 and $p-3$ being non negative. We found a pair of non negative integers $(m', p') = (4, p-3)$ such that $k+1 = 4m' + 5p'$. P(k+1) is therefore true.

In both cases, $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n \geq 12$.