Discussion

Exercise 1

Show that any postage value $n \geq 20$ can be made with only 5-cent and 6-cent stamps.

$$P(n): \quad n = 5 \times a(n) + 6 \times b(n)$$

$a(n)$ and $b(n)$ are integers.
$a(n) \geq 0$ and $b(n) \geq 0$

I want to show that $P(n)$ is true, $n \geq 20$.

Proof by induction:

basis step: $n = 20$

$P(20)$: $\quad 20 = 5 \times 4 + 6 \times 0$

$a(20) = 4$
$b(20) = 0$
) positive integers.

$P(20)$ is true.

Inductive step: $P(n) \to P(n+1)$, $n \geq 20$. ②

I assume $P(n)$ is true,

$$n = 5 \times a(n) + 6 \times b(n)$$

$a(n), b(n)$ positive integers.

$P(n+1)$?

$$\begin{aligned} n+1 &= 5 \times a(n) + 6 \times b(n) + 1 \\ &= 5 \times a(n) + 6 \times b(n) + 6 - 5 \\ &= 5 \times \underbrace{(a(n) - 1)}_{\substack{\text{integer,} \\ \text{but could be} \\ \text{negative.}}} + 6 \times \underbrace{(b(n) + 1)}_{\text{positive integer}} \end{aligned}$$

Case 1: $a(n) > 0$

I can define

$a(n+1) = a(n) - 1$ positive integer

$b(n+1) = b(n) + 1$ //

$n+1 = 5 \times a(n+1) + 6 \times b(n+1)$

Case 2 $a(n) = 0$

$n = 6 \times b(n)$

$n+1 = 6 \times b(n) + 1$

$n+1 = 6 \times b(n) + 25 - 24$

$$n+1 = 6\times b(n) + 5\times 5 - 6\times 4$$
$$= \underbrace{5\times 5}_{\text{positive integer}} + 6\times(\underline{b(n)-4})$$

We know $n \geq 20$

$$6\times b(n) \geq 20$$
$$b(n) \geq \frac{20}{6}$$

Since $b(n)$ is an integer, $b(n) \geq 4$

I can define:
$$a(n+1) = 5 \quad \text{positive integer.}$$
$$b(n+1) = b(n) - 4 \quad '' \quad ''$$

$P(n+1)$ is true

The method of proof by induction allows me to conclude that $P(n)$ is true, $\forall n \geq 20$

Exercise 2:

Let $F(n)$ be the Fibonacci numbers.

Show that $F(n-1)\,F(n+1) - F(n)^2 = (-1)^n \quad n > 1.$

Let $A(n) = F(n-1)\,F(n+1) - F(n)^2$

$B(n) = (-1)^n$

$P(n)$: $A(n) = B(n)$ when $n > 1$.

Proof by induction:

Basis step: $n = 2$

$A(2) = F(1)\,F(3) - F(2)^2 = 1 \times 2 - 1 = 1$

$B(2) = (-1)^2 = 1$

$A(2) = B(2)$: $P(2)$ is true.

Inductive step: $P(n) \to P(n+1) \quad n > 1.$

I assume $P(n)$ is true, $A(n) = B(n)$

$$A(n) = F(n+1)\,F(n-1) - F(n)^2$$ 5

$$B(n) = (-1)^n$$

$$B(n+1) = (-1)^{n+1} = -(-1)^n = -B(n)$$

$$A(n+1) = F(n+1+1)\,F(n+1-1) - F(n+1)^2$$
$$= F(n+2)\,F(n) - F(n+1)^2$$

I know $F(n+2) = F(n+1) + F(n)$

$$A(n+1) = \big(F(n+1) + F(n)\big)\,F(n) - F(n+1)^2$$
$$= F(n+1)\,F(n) + F(n)^2 - F(n+1)^2$$
$$= F(n)^2 + F(n+1)\,F(n) - F(n+1)^2$$
$$= F(n)^2 + F(n+1)\,[F(n) - F(n+1)]$$

$$F(n+1) = F(n) + F(n-1)$$
$$\rightarrow F(n) - F(n+1) = -F(n-1)$$

$$A(n+1) = F(n)^2 - F(n-1)\,F(n+1)$$
$$= -A(n) = -B(n) = B(n+1)$$

$P(n+1)$ is true.

The method of proof by induction allows me to conclude that $P(n)$ is true, $n > 1$.