Exercise 1

Let n be an integer.

Show that $\underbrace{n \text{ is even}}_{p}$ if and only if $\underbrace{3n+5 \text{ is odd}}_{q}$.

p: n is even

q: $3n+5$ is odd.

As I know that

$$p \leftrightarrow q \iff (p \to q) \wedge (q \to p)$$

I need to show $(p \to q)$ AND $(q \to p)$

Step 1: show $p \to q$.

p: n is even → $\neg p$: n is odd

q: $3n+5$ is odd ← $\neg q$: $3n+5$ is even.

Direct proof: p is true, n is even.

There exists an integer k such that $n = 2k$.

$$3n+5 = 3(2k)+5$$
$$= 6k+4+1$$
$$= 2[\underbrace{3k+2}_{\text{integer}}]+1$$

Therefore $3n+5$ is odd: q is true.

Step 2: Prove $q \to p$.

q: $3n+5$ is odd
p: n is even
$\neg q$: $3n+5$ is even
$\neg p$: n is odd

We do an indirect proof:

We assume $\neg p$ is true: n is odd.

There exists an integer k such that $n = 2k+1$.

$$3n+5 = 3(2k+1)+5$$
$$= 6k+3+5$$
$$= 6k+8$$
$$= 2[\underbrace{3k+4}_{\text{integer}}].$$

Therefore $3n+5$ is even: $\neg q$ is true.

I have shown $\neg p \to \neg q$, which also mean $q \to p$

Exercise 2

Let a and b be 2 integers.

Show that if $a+b$ is even, then a^2+b^2 is even.

a) Identify the problem: proving an implication.

p : $a+b$ is even $\quad \neg p$: $a+b$ is odd

q : a^2+b^2 is even $\quad \neg q$: a^2+b^2 is odd

b) Choose method of proof.

I use a direct proof

c) Body of the truth:

I assume p is true : $a+b$ is even.

There exists an integer k such that $a+b=2k$.

$$a+b = 2k$$
$$(a+b)^2 = 4k^2$$
$$a^2+b^2+2ab = 4k^2$$

$$a^2 + b^2 + 2ab = 4k^2$$

$$a^2 + b^2 = 4k^2 - 2ab$$

$$= 2[\underbrace{2k^2 - ab}_{\text{integer}}]$$

$a^2 + b^2$ is even : q is true.

Conclusion: I have shown $p \to q$ using a direct proof.

Exercise 3 We are on the island of knights and knaves. We meet Alex, John, and Sally.

Alex: "If Sally is a knight, then I am a knave."

John: "Alex and Sally are of the same type"

Sally: "I like chocolate".

K: knight. N: knave

Alex	John	Sally	S_1	S_2	
K	K	K ✓	F	T	
K	K	N	T	F	X
K	N	K	F	T	X
K	N	N	T	F	
N	K	K	F	F	X
N	K	N	T	T	X
N	N	K	F	F	X
N	N	N	F	T	X

if Sally is a knight, Alex is a knave
p ↓ q

The only option that remains is that Alex is knight while John and Sally are knaves. Therefore Sally does not like chocolate.