Discussion 2/21/25

Exercise 1

Let n be an integer. Show that if $3n$ is odd, then $(n+1)^2$ is even.

(1) Identify the problem

We need to prove $p \to q$

p: $3n$ is odd $\neg p$: $3n$ is even

q: $(n+1)^2$ is even $\neg q$: $(n+1)^2$ is odd

(2) which method of proof do I use?

Direct proof: Premise: p is true.

(3) Body of proof

$3n$ is odd.

$3n = 2k+1$, k integer.

$n + 2n = 2k+1$

$n = 2\underbrace{(k-n)}_{\text{integer}} + 1$

n is odd.

$n = 2l+1$, l integer.

$$(n+1)^2 = (2p+1+1)^2$$
$$= (2p+2)^2$$
$$= 2 \times \underbrace{[2 \times (p+1)^2]}_{\text{integer}}$$

$(n+1)^2$ is even : q is true

4 Conclusion

By direct proof, I have shown $p \to q$.

Exercise 2

Let x be a real number, different from 0.
Show a proof by contradiction to show that if $x + \frac{1}{x} < 2$, then $x < 0$.

1 Identify the problem

We need to prove $p \to q$

$p: \; x + \frac{1}{x} < 2$ $\qquad$ $\neg p: \; x + \frac{1}{x} \geq 2$

$q: \; x < 0$ $\qquad$ $\neg q: \; x \geq 0$.

2 Method of proof : contradiction.

Premise: p is true and $\neg q$ is true.

3 Body of the proof

p is true: $x + \frac{1}{x} < 2$ (1)

$\neg q$ is true: $x > 0$.

I multiply (1) with x:

$$x\left(x + \frac{1}{x}\right) < 2x$$

$$x^2 + 1 < 2x$$

$$x^2 - 2x + 1 < 0$$

$$(x-1)^2 < 0$$

This is a contradiction.

4 Conclusion:

By proof by contradiction, I have shown $p \to q$ is true.

Exercise 3

Let a and b be 2 integers.

Show that $a^2 - 4b \neq 2$

1) Identify the problem

I want to show p is true

p: $a^2 - 4b \neq 2$

2) Method of proof

Proof by contradiction.

Premise: $\neg p$ is true.

3) Body of the proof:

$$a^2 - 4b = 2$$

$$a^2 = 2 + 4b$$

$$= 2\underbrace{(1+2b)}_{\text{integer}}.$$

a^2 is even $\longrightarrow$ a is even.

There exists an integer k such that

$$a = 2k.$$

$$a^2 - 4b = 2$$

$$\hookrightarrow (2k)^2 - 4b = 2$$

$$\hookrightarrow 4k^2 - 4b = 2$$

$$\div 2 \quad \hookrightarrow 2k^2 - 2b = 1$$

$$LHS = 2k^2 - 2b = 2(\underbrace{k^2 - b}_{\text{integer}}) \rightarrow \text{even}$$

$$RHS = 1 \rightarrow \text{odd.}$$

This is a contradiction.

(4) <u>Conclusion:</u>

By proof by contradiction, I have shown p is true.

Let n be a natural number.

Show that $\dfrac{6n+3}{4n+6}$ is not an integer.

① Identify the problem.

I want to show p is true:

p: $\dfrac{6n+3}{4n+6}$ is not an integer.

② Method of proof

Proof by contradiction.

Premise: p is false

③ Body of the proof

$\dfrac{6n+3}{4n+6}$ is an integer.

Let k be that integer.

$$\frac{6n+3}{4n+6} = k$$

$$\underbrace{6n+3}_{LHS} = \underbrace{k(4n+6)}_{RHS}$$

$$\begin{aligned} LHS &= 6n + 3 \\ &= 6n + 2 + 1 \\ &= 2\underbrace{(3n+1)}_{\text{integer}} + 1 \end{aligned}$$

LHS is odd

$$\begin{aligned} RHS &= k(4n+6) \\ &= 2\left[\underbrace{k(2n+3)}_{\text{integer}}\right] \end{aligned}$$

RHS is even

I have reached a contradiction.

(4) Conclusion

By proof by contradiction, p is true.