Discussion 2/26

Proofs by induction.

Exercise 1

Show that

$$\forall n \in \mathbb{N}, \quad \sum_{i=1}^{n} (-1)^i i^2 = \frac{(-1)^n n(n+1)}{2}$$

Definition of the problem:

let $A(n) = \sum_{i=1}^{n} (-1)^i i^2$

$B(n) = \frac{(-1)^n n(n+1)}{2}$

$P(n): \quad A(n) = B(n)$

Method of proof

Induction : I need to show:

Basis step: $P(1)$ is true

Inductive step: $P(n) \rightarrow P(n+1)$ is true, $n \geq 1$

~~Body of the proof:~~

Basis step: $P(1)$ is true.

$$A(1) = \sum_{i=1}^{1} (-1)^i i^2 = (-1) \times 1^2 = -1$$

$$B(1) = \frac{(-1) \times 1 (1+1)}{2} = -1$$

$\Rightarrow$ $P(1)$ is true

Inductive step:

Premise: $P(n)$ is true. $A(n) = B(n)$

$$A(n+1) = \sum_{i=1}^{n+1} (-1)^i i^2$$

$$= \sum_{i=1}^{n} (-1)^i i^2 + (-1)^{n+1}(n+1)^2$$

$$= A(n) + (-1)^{n+1}(n+1)^2$$

$$= B(n) + (-1)^{n+1}(n+1)^2$$

$$= \frac{(-1)^n n(n+1)}{2} + (-1)^{n+1}(n+1)^2$$

$$= \frac{(-1)^n n(n+1) + 2(-1)^{n+1}(n+1)^2}{2}$$

$$= \frac{-(-1)^{n+1} n(n+1) + 2(-1)^{n+1}(n+1)^2}{2}$$

$$= \frac{(-1)^{n+1}(n+1)\left[-n + 2n+2\right]}{2} = \frac{(-1)^{n+1}(n+1)(n+2)}{2}$$

$$A(n+1) = \frac{(-1)^{n+1}(n+1)(n+2)}{2}$$

$$B(n+1) = \frac{(-1)^{n+1}(n+1)(n+2)}{2}$$

$\Rightarrow$ $P(n+1)$ is true

<u>Conclusion:</u> The method of proof by induction allows me to conclude that $P(n)$ is true $\forall n \in \mathbb{N}$

<u>Exercise 2</u> Any postage value n can be made with only 4 cts and 5 cts stamps, when $n \geq 15$

<u>Definition of the problem:</u>

$P(n)$: $n = 4a_n + 5b_n$

- a_n and b_n are integers
- $a_n \geq 0$, $b_n \geq 0$

$P(n)$ is true when $n \geq 15$.

Proof by induction:

Basis step : $P(15)$ is true

Inductive step: $P(n) \rightarrow P(n+1) \quad n \geq 15$

Body of the proof:

Basis step:

$$15 = 4 \times \underbrace{0}_{a_{15}} + 5 \times \underbrace{3}_{b_{15}}$$

$$16 = 4 \times \underbrace{4}_{a_{16}} + 5 \times \underbrace{0}_{b_{16}}$$

$$17 = 4 \times 3 + 5 \times 1$$

$$43 = 4 \times 2 + 5 \times 7$$

$$\cancel{= 4 \times 12 + 5 \times 1}$$

$P(15)$ is true.

Inductive step: $P(n) \Rightarrow P(n+1) \quad n \geq 15$

Premise:

there exists 2 integers a_n, b_n with $a_n \geq 0, \quad b_n \geq 0$

$$n = 4a_n + 5b_n$$

$$n+1 = 4a_n + 5b_n + 1$$

$$n+1 = 4a_n + 5b_n + 5 - 4$$

$$n+1 = 4(a_n - 1) + 5(b_n + 1)$$

$a_n - 1$ and $b_n + 1$ are integers.

$b_n + 1$ is positive.

$a_n - 1$?

Case 1 : $a_n - 1 \geq 0$

$$a_{n+1} = a_n - 1 \geq 0 \Rightarrow P(n+1) \text{ is true.}$$

$$b_{n+1} = b_n + 1 \geq 0$$

Case 2 $a_n - 1 < 0 \to a_n < 1 \to a_n = 0$

$$n = 4a_n + 5b_n$$

$$n = 5b_n$$

$$n+1 = 5b_n + 1$$

$$= 5b_n + 16 - 15$$

$$= 4 \times 4 + 5(b_n - 3)$$

$a_{n+1} = 4 \geq 0$ $\qquad n \geq 15$

$b_{n+1} = b_n - 3 \geq 0$ $\qquad 5b_n \geq 15 \to b_n \geq 3$

$P(n+1)$ is true

Conclusion: The method of proof by induction allows me to conclude that $P(n)$ is true, $n \geq 15$.