To prove a property $P(n)$ is true $\forall n \geq n_0$

- Basis step: $P(n_0)$ is true
- Inductive step: $P(n) \rightarrow P(n+1)$ $\forall n \geq n_0$

Example:

Show that $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$, $\forall n \in \mathbb{N}$.

Defining the problem:

Let us define:

$$A(n) = \sum_{i=1}^{n} \frac{1}{i(i+1)}$$

$$B(n) = \frac{n}{n+1}$$

$$P(n): \quad A(n) = B(n)$$

I want to prove $P(n)$ is true, $\forall n \geq 1$

Method of proof:

Induction

Basis step: $P(1)$ is true

Inductive step: $P(n) \to P(n+1)$ $n \geq 1$

Body of the proof:

Basis step:

$$A(1) = \sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = \frac{1}{2}$$

$$B(1) = \frac{1}{1+1} = \frac{1}{2}$$

$A(1) = B(1) \to P(1)$ is true.

Inductive step: $P(n) \to P(n+1)$ $n \geqslant 1$

Premise: $P(n)$ is true: $A(n) = B(n)$

$$A(n+1) = \sum_{i=1}^{n+1} \frac{1}{i(i+1)}$$

$$= \sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+2)}$$

$$= A(n) + \frac{1}{(n+1)(n+2)}$$

$$= B(n) + \frac{1}{(n+1)(n+2)}$$

$$A(n) = \sum_{i=1}^{n} \frac{1}{i(i+1)}$$

$$= \frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \frac{1}{3(3+1)} + \cdots \frac{1}{n(n+1)}$$

$$A(n+1) = \sum_{i=1}^{n+1} \frac{1}{i(i+1)}$$

$$= \frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \frac{1}{3(3+1)} + \cdots \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)}$$

$$A(n+1) = B(n) + \frac{1}{(n+1)(n+2)}$$

$$= \frac{n}{(n+1)} + \frac{1}{(n+1)(n+2)}$$

$$= \frac{n(n+2)}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)}$$

$$= \frac{n(n+2)+1}{(n+1)(n+2)}$$

$$= \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)}$$

$$= \frac{n+1}{n+2}$$

$$B(n+1) = \frac{n+1}{n+1+1} = \frac{n+1}{n+2}$$

$$A(n+1) = B(n+1) \rightarrow P(n+1) \text{ is true}$$

Conclusion: the method of proof by induction allows me to conclude that $P(n)$ is true, $\forall n \geq 1$

Exercise 2

Any postage value n can be made with a set of 4 cents and 5 cents stamps, when $n \geq 15$.

Definition of the problem:

$$P(n): \begin{cases} \text{There exists } a_n \text{ and } b_n \text{ such that} \\ n = 4a_n + 5b_n \\ a_n \text{ integer} \quad a_n \geq 0 \\ b_n \text{ integer} \quad b_n \geq 0 \end{cases}$$

Method of proof:

Induction

Basis step: $P(15)$ is true

Inductive step: $P(n) \to P(n+1)$ $n \geq 15$

Body of the proof:

Basis ~~Induction~~ step:

$$15 = 4 \times \underbrace{0}_{a_{15}} + 5 \times \underbrace{3}_{b_{15}}$$

$P(15)$ is true.

Inductive step: $P(n) \to P(n+1) \quad n \geq 15$

Premise: $P(n)$ is true.

There exists $a_n \geq 0$ (integer), $b_n \geq 0$ (integer) such that $n = 4a_n + 5b_n$

$$n+1 = 4a_n + 5b_n + 1$$

$$= 4a_n + 5b_n + 5 - 4$$

$$n+1 = 4(a_n - 1) + 5(b_n + 1)$$

Case 1 $\quad a_n - 1 \geq 0 \quad (a_n \geq 1)$

$$a_{n+1} = a_n - 1 \geq 0$$

$$b_{n+1} = b_n + 1 \geq 0$$

$$n+1 = 4a_{n+1} + 5b_{n+1} \quad : P(n+1) \text{ is true.}$$

Case 2 : $\quad a_n = 0$.

$$n = 5b_n$$

$$n+1 = 5b_n + 1$$

$$= 5b_n + 16 - 15$$

$$n+1 = 5b_n + 16 - 15$$

$$= 4 \times 4 + 5(b_n - 3)$$

$n = 5b_n$ and $n \geqslant 15$

$5b_n \geqslant 15 \rightarrow \underline{b_n \geqslant 3}$

$a_{n+1} = 4 \geqslant 0$

$b_{n+1} = b_n - 3 \geqslant 0$ $\quad P(n+1)$ is true.

Conclusion: The method of proof by induction allows me to conclude $P(n)$ is true $\forall n \geqslant 15$.