Exercise 1

Show that n^2+n+1 is odd for all natural numbers n

a) Using a direct proof

b) Using a proof by induction.

A) Direct proof

Let n be an natural number.

$n^2+n = n(n+1)$ is always even (prayer)

There exists an integer k such that

$$n^2+n = 2k$$

$$n^2+n+1 = 2\underbrace{k}_{\text{integer}}+1$$

Therefore n^2+n+1 is odd.

B) <u>Proof by Induction</u>

$P(n)$: n^2+n+1 is odd.

I want to show $P(n)$ is true, $\forall n \geq 1$.

a) <u>Basis step:</u> $n=1$

~~n^2+n+1~~

Let us define $A(n) = n^2+n+1$

$A(1) = 1+1+1 = 3$ odd.

$P(1)$ is true.

b) <u>Inductive step:</u> $P(n) \to P(n+1)$ $n \geq 1$.

We assume $P(n)$ is true:

$A(n) = n^2+n+1$ is odd.

There exists an integer k such that

$A(n) = n^2+n+1 = 2k+1$

$$A(n+1) = (n+1)^2 + n+1+1$$
$$= n^2+2n+1+n+2$$
$$= n^2+n+1+2n+2$$
$$= 2k+1+2n+2$$
$$= 2(\underbrace{k+n+1}_{\text{integer}})+1$$

$A(n+1)$ is odd : $P(n+1)$ is true.

The method of proof by induction allows me to conclude that $P(n)$ is true, $\forall n \geq 1$

Exercise 2 Let F_n be the Fibonacci numbers. Show that

F_{5n} is a multiple of 5, $\forall n \geq 1$.

Proof by induction:

$P(n)$: F_{5n} is a multiple of 5

I want to show $P(n)$ is true, $\forall n \geq 1$.

Basis step: $n = 1$

$F_{5\times 1} = F_5$

n	F_n
0	$F_0 = 0$
1	$F_1 = 1$
2	$F_2 = 1$
3	$F_3 = 2$
4	$F_4 = 3$
5	$F_5 = 5$

F_5 is a multiple of 5.

Inductive step: $P(n) \longrightarrow P(n+1)$ $n \geq 1$.

I assume $P(n)$ is true.

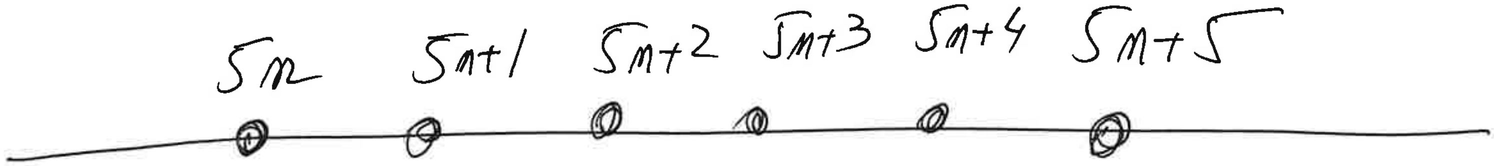
F_{5n} is a multiple of 5

There exists an integer k such that

$$F_{5n} = 5k$$

$$F_{5(n+1)} = F_{5n+5}$$

$$= F_{5n+4} + F_{5n+3}$$



$$F_{5n+5} = F_{5n+4} + F_{5n+3}$$

$$= F_{5n+3} + F_{5n+2} + F_{5n+3}$$

$$= 2F_{5n+3} + F_{5n+2}$$

$$= 2F_{5n+2} + 2F_{5n+1} + F_{5n+2}$$

$$= 3F_{5n+2} + 2F_{5n+1}$$

$$= 3(F_{5n+1}) + 3F_{5n} + 2F_{5n+1}$$

$$= 5F_{5n+1} + 3 \times 5k$$

$$= 5(F_{5n+1} + 3k) \rightarrow \text{multiple of } 5$$

$P(n+1)$ is true.

The method of proof by induction allows me to conclude that $P(n)$ is true, $\forall n \geq 1$.

Exercise 3

Show that $n+3 = n+7 \quad \forall n \geq 1$,

$P(n): \quad n+3 = n+7$

Inductive step: $\quad P(n) \to P(n+1) \quad \forall n \geq 1$

I assume $P(n)$ is true.

$$n+3 = n+7$$

$A(n) = n+3$,

$B(n) = n+7$

$P(n): \quad A(n) = B(n)$

$A(n+1) = n+1+3 = n+4 = A(n)+1$

$B(n+1) = n+1+7 = n+7+1$

$= B(n)+1$

$= A(n)+1$

$A(n+1) = B(n+1) \qquad P(n+1)$ is true.

However, we cannot prove the basis step!!

Exercise 4

Show that $2^n < n!$ $n \geq 4$.

Let $P(n): \quad 2^n < n!$

We want to show $P(n)$ is true when $n \geq 4$

Basis step: $n = 4$

$A(n) = 2^n$

$B(n) = n!$

$A(4) = 2^4 = 16$

$B(4) = 4! = 4 \times 3 \times 2 \times 1 = 24$

$A(4) < B(4)$ $\therefore$ $P(4)$ is true.

Inductive step: $P(n) \rightarrow P(n+1)$, $n \geq 4$.

I assume $P(n)$ is true:

$A(n) < B(n)$

$2^n < n!$

$\hookrightarrow 2^{n+1} < 2n!$

$2 < n+1$

$2n! < (n+1)n!$

$2^{n+1} < 2n! < (n+1)!$

$P(n+1)$ is true.

$2^{n+1} < (n+1)!$

$(n+1)n!$