Discussion 3/7

Fibonacci

Fibonacci numbers: F_i

$F_{\phi} = 0$

$F_1 = 1$

$F_2 = 1$

$F_3 = 2$

$F_n = F_{\cancel{n}-1} + F_{n-2} \quad n \geq 2$

Exercise 1:

Let F_n be the Fibonacci numbers.

Show that

$$\sum_{i=0}^{n} F_{2i} = F_{2n+1} - 1 \quad \forall n \geq 0.$$

Definition of the problem:

$A(n) = \sum_{i=0}^{n} F_{2i}$

$B(n) = F_{2n+1} - 1$

$P(n): \quad A(n) = B(n)$

$A(n) = \sum_{i=0}^{n} F_{2i}$ ②

$B(n) = F_{2n+1} - 1$

I want to show:

$$P(n): \quad A(n) = B(n)$$

is true for all $n \geq 0$

Proof by induction

Basis step $\qquad P(0)$ is true?

$A(0) = \sum_{i=0}^{0} F_{2i} = F_{2\times\varnothing} = F_{\varnothing} = 0$

$B(0) = F_1 - 1 = 1 - 1 = \varnothing$

> P(0) is true.

Inductive step:

$$P(n) \to P(n+1) \qquad n \geq 0.$$

Premise: $P(n)$ is true, $A(n) = B(n)$.

$$A(n+1) = \sum_{i=0}^{n+1} F_{2i} = \sum_{i=0}^{n} F_{2i} + F_{2(n+1)}$$

$$= A(n) + F_{2n+2} = B(n) + F_{2n+2}$$

$$= F_{2n+1} - 1 + F_{2n+2} = F_{2n+1} + F_{2n+2} - 1$$

$$\cancel{B(n+1)} = F_{2n+3} - 1$$

$$B(n+1) = F_{2(n+1)+1} - 1 = F_{2n+3} - 1$$

$A(n+1) = B(n+1) \to P(n+1)$ is true

The method of proof by induction allows me to conclude that $P(n)$ is true, $\forall n \geq 0$.

Exercise 2

~~Show that there exists an integer~~

Show that for each $n \geq 0$, there exists an integer m such that

$$F_{4n} = 3m,$$

Proof by induction:

Define: $P(n)$: There exists an integer m such that $F_{4n} = 3m$

Basis step: is $P(\emptyset)$ true?

Can you find an integer m such that

$$F_{\emptyset} = 3m \;?$$

$$\emptyset = 3 \times \emptyset$$

$m = \emptyset$ in this case. $P(\emptyset)$ is true.

Inductive step:

show that $P(n) \rightarrow P(n+1)$ $\quad n \geq 0$

Premise: $P(n)$ is true.

I know there exists an integer m such that

$$F_{4n} = 3m.$$

is $P(n+1)$ true?

$$
\begin{aligned}
F_{4(n+1)} &= F_{4n+4} \\
&= F_{4n+3} + F_{4n+2} \\
&= F_{4n+2} + F_{4n+1} + F_{4n+1} + F_{4n} \\
&= F_{4n+1} + F_{4n} + F_{4n+1} + F_{4n+1} + F_{4n} \\
&= 3\,F_{4n+1} + 2\,F_{4n} \\
&= 3\,F_{4n+1} + 2 \times 3 \times m \\
&= 3\,(\underbrace{F_{4n+1} + 2m}_{\text{integer}: \ p})
\end{aligned}
$$

$P(n+1)$ is true.

The method of proof by induction allows me to conclude that $P(n)$ is true for all $n \geq 0$.