Discussion 2/21

Exercise 1

Let n be an integer.
Use 3 methods of proof (direct, indirect, and contradiction) to show that if $\underbrace{5n^2+2 \text{ is even}}_{p}$ then $\underbrace{n \text{ is even}}_{q}$.

p: $5n^2+2$ is even $\qquad \neg p$: $5n^2+2$ is odd

q: n is even $\qquad \neg q$: n is odd.

Indirect proof $\neg q \to \neg p$

Assumption: $\neg q$ is true. n is odd. There exists an integer k such that

$$n = 2k+1$$

$$5n^2+2 = 5(2k+1)^2 + 2$$
$$= 5(4k^2+4k+1)+2 = 20k^2+20k+7$$
$$= 2(10k^2+10k+3)+1$$

$10k^2 + 10k + 3$ is an integer.

Therefore $5n^2 + 2$ is odd $\therefore \neg p$ is true.

Direct proof $p \to q$

I assume p is true.

$5n^2 + 2 = 2k$, k is an integer.

$$n^2 + 4n^2 + 2 = 2k$$

$$n^2 = 2k - 4n^2 - 2$$

$$= 2(\underbrace{k - 2n^2 - 1}_{\text{integer}})$$

n^2 is even $\to$ n is even.

Proof by contradiction. Assume $(p \to q)$ is false is the same thing as assuming that p is true and q is false.

p is true: $5n^2+2$ is even

q is false: n is odd. There exists an integer k such that $n = 2k+1$

$$5n^2+2 = 2(\underbrace{10k^2+10k+3}_{\text{integer}}) + 1$$

contradiction

$5n^2+2$ is odd

③

Exercise 2 let a and b be two integers.

Show that if $\underbrace{a^2(b^2-2b) \text{ is odd}}_{p}$, then $\underbrace{a \text{ is odd and } b \text{ is odd}}_{q}$.

p: $a^2(b^2-2b)$ is odd $\quad$ $\neg p$: $a^2(b^2-2b)$ is even

q: a is odd and b is odd $\quad$ $\neg q$: a is even or b is even.

Indirect proof: $\neg q \to \neg p$

$\neg q \iff \neg q_1 \vee \neg q_2$. I need to show:

Case 1 $\neg q_1 \to \neg p$
a is even: there exists an integer k such that $a = 2k$

$$a^2(b^2-2b) = 4k^2(b^2-2b) = 2\underbrace{\left[2k^2(b^2-2b)\right]}_{\text{integer}}$$

therefore $a^2(b^2-2b)$ is even; $\neg p$ is true.

Case 2: $\neg q_2 \to \neg p$
b is even: there exists an integer k such that $b = 2k$

$$a^2(b^2-2b) = a^2(4k^2-4k) = 2\underbrace{\left[a^2(2k^2-2k)\right]}_{\text{integer}}$$

$a^2(b^2-2b)$ is even; $\neg p$ is true.

Exercise 3

Let x be a real number, such that $x \neq 0$.
Use a proof by contradiction to show that if $x + \frac{1}{x} < 2$ then $x < 0$

p $\qquad$ q

$p: \ x + \frac{1}{x} < 2$ $\qquad$ $\neg p: \ x + \frac{1}{x} \geq 2$

$q: \ x < 0$ $\qquad$ $\neg q \ \ x \geq 0$

Proof by contradiction: We assume p is true and $\neg q$ is true

$$x + \frac{1}{x} < 2$$

$$x \geq 0 \ (\text{in fact, } x > 0)$$

$$x + \frac{1}{x} < 2$$

$\times x$

$$x^2 + 1 < 2x$$

$$x^2 - 2x + 1 < 0$$

$$(x-1)^2 < 0 \quad \text{contradiction!}$$