Discussion : 2/26

Induction.

Exercise 1

Show that

$$\sum_{i=1}^{n} (i+1) 2^i = n \, 2^{n+1}, \text{ for all natural numbers.}$$

Definitions:

$$A(n) = \sum_{i=1}^{n} (i+1) 2^i = 2 \times 2 + 3 \times 2^2 + \cdots (n+1) 2^n$$

$$B(n) = n \, 2^{n+1}$$

$$P(n): \quad A(n) = B(n)$$

To prove $P(n)$ is true for all n, I use a proof by induction.

Basis step: is $P(1)$ true?

$$\left.\begin{aligned} A(1) &= 2 \times 2 = 4 \\ B(1) &= 1 \times 2^2 = 4 \end{aligned}\right\} \; P(1) \text{ is true.}$$

Inductive step: $p(n) \to p(n+1)$ is true $\forall n \geq 1$

I assume $P(n)$ is true: $A(n) = B(n)$

what about $P(n+1)$?

$$A(n+1) = \sum_{i=1}^{n+1} (i+1)2^i = \underbrace{2\times 2 + \cdots\ (n+1)2^n}_{} + (n+2)2^{n+1}$$

$$\begin{aligned} A(n+1) &= A(n) + (n+2)2^{n+1} \\ &= B(n) + (n+2)2^{n+1} \\ &= n\,2^{n+1} + (n+2)2^{n+1} \\ &= (n+n+2)2^{n+1} \\ &= 2(n+1)2^{n+1} \\ &= (n+1)2^{n+2} \end{aligned}$$

$$B(n+1) = (n+1)2^{n+2}$$

therefore, $A(n+1) = B(n+1)$: $P(n+1)$ is true.

The method of proof by induction allows me to conclude that $P(n)$ is true for all natural numbers.

Exercise 2

Show that
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
for all natural numbers n.

Definition:
$$A(n) = \sum_{i=1}^{n} i^2 = 1 + 4 + 9 + \dots n^2$$
$$B(n) = \frac{n(n+1)(2n+1)}{6}$$

$$P(n): \quad A(n) = B(n)$$

We use a proof by induction:

is $P(1)$ true?

Basis step:
$$A(1) = 1^2 = 1$$
$$B(1) = \frac{1 \times (1+1)(2+1)}{6} = \frac{2 \times 3}{6} = 1$$
$P(1)$ is true.

<u>Inductive step:</u> $P(n) \to P(n+1)$ is true for all $n \geq 1$.

We assume $P(n)$ is true: $A(n) = B(n)$

$$A(n+1) = \underbrace{1 + 4 + \cdots n^2}\, + (n+1)^2$$

$$\begin{aligned}
A(n+1) &= A(n) + (n+1)^2 \\
&= B(n) + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6} \\
&= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\
&= \frac{(n+1)\left[n(2n+1) + 6(n+1)\right]}{6} \\
&= \frac{(n+1)\left[2n^2 + 7n + 6\right]}{6}
\end{aligned}$$

$$\begin{aligned}
B(n+1) &= \frac{(n+1)(n+2)(2n+3)}{6} \\
&= \frac{(n+1)\left[2n^2 + 7n + 6\right]}{6}
\end{aligned}$$

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Therefore $A(n+1) = B(n+1)$: $P(n+1)$ is true

The method of proof by induction allows me to conclude that $P(n)$ is true for all natural numbers n.

Can you make any postage values greater or equal to 8 with only 3c and 5c stamps?

P(n) : $n = 3a_n + 5b_n$

a_n and b_n need to be positive integers.

P(n) is true for all $n \geq 8$.

Proof by induction:

Basis step: Is P(8) true?

$$8 = 3 + 5$$

We define: $a_8 = b_8 = 1$

Since a_8 and b_8 are positive integers, P(8) is true.

Inductive step: $P(n) \to P(n+1)$ is true for all $n \geq 8$

I assume $P(n)$ is true:
there exist 2 positive integers a_n and b_n such that

$$n = 3a_n + 5b_n$$

$$\begin{aligned} n+1 &= 3a_n + 5b_n + 1 \\ &= 3a_n + 5b_n + 6 - 5 \\ &= 3a_n + 5b_n + 2\times 3 - 5 \\ &= 3(a_n + 2) + 5(b_n - 1) \end{aligned}$$

2 cases:

Case 1: $b_n > 0$

$$\left.\begin{aligned} a_{n+1} &= a_n + 2 > 0 \\ b_{n+1} &= b_n - 1 > 0 \end{aligned}\right\} \text{positive integers}$$

and $n+1 = 3a_{n+1} + 5(b_{n+1})$ $\therefore P(n+1)$ is true.

Case 2: $b_n = 0$

$$n = 3a_n$$

$$\begin{aligned} n+1 &= 3a_n + 1 \\ &= 3a_n + 10 - 9 \\ &= 3(a_n - 3) + 2 \times 5 \end{aligned}$$

$n \geq 8$

$3a_n \geq 8 \rightarrow$ therefore $a_n \geq 3$

$a_{n+1} = a_n - 3 \geq 0$

$b_{n+1} = 2 \geq 0$

and $n+1 = 3a_{n+1} + 5b_{n+1}$

Therefore $P(n+1)$ is true.

The method of proof by induction allows me to conclude that $P(n)$ is true for all $n \geq 8$.