Discussion 2/28

Induction

Exercise 1:

Show that $\sum_{i=1}^{n} (i+1)2^{i} = n\,2^{n+1}$

for all natural numbers n.

Definitions:

$$A(n) = \sum_{i=1}^{n} (i+1)2^{i}$$
$$= 2\times 2 + 3\times 2^{2} + 4\times 2^{3} + \cdots (n+1)2^{n}$$

$$B(n) = n\,2^{n+1}$$

$$P(n): \quad A(n) = B(n)$$

Proof by induction:

- **Basis step:** $P(1)$ is true?

$A(1) = 2\times 2 = 4$

$B(1) = 1\times 2^{1+1} = 4$

$>$ $P(1)$ is true.

Inductive step: We want to show

$$P(n) \to P(n+1) \text{ for all } n.$$

Direct method:

We assume $P(n)$ is true: $A(n) = B(n)$

$$A(n+1) = 2\times 2 + 3\times 2^2 + \cdots\cdots (n+1)2^n + (n+2)2^{n+1}$$
$$= A(n) + (n+2)2^{n+1}$$
$$= B(n) + (n+2)2^{n+1}$$
$$= n\,2^{n+1} + (n+2)\,2^{n+1}$$
$$= (n+n+2)\,2^{n+1} = (2n+2)2^{n+1} = (n+1)2^{n+2}$$

$$B(n+1) = (n+1)2^{n+2}$$

$A(n+1) = B(n+1)$: $P(n+1)$ is true.

The method of proof by induction allows me to conclude that $P(n)$ is true for all natural numbers n.

Exercise 2

Show that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

for all natural numbers n.

Definitions:

$$A(n) = \sum_{i=1}^{n} i^2 = 1 + 4 + 9 + \cdots n^2$$

$$B(n) = \frac{n(n+1)(2n+1)}{6}$$

$$P(n): \quad A(n) = B(n)$$

Proof by induction:

Basis step: show $P(1)$ is true

$$\left.\begin{aligned} A(1) &= 1 \\ B(1) &= \frac{1 \times (1+1)(2+1)}{6} = \frac{2 \times 3}{6} = 1 \end{aligned}\right\} P(1) \text{ is true}$$

Inductive step:

$P(n) \to P(n+1)$ is true for all n, natural number?

Direct proof: I assume $P(n)$ is true,
$A(n) = B(n)$

$$A(n+1) = 1 + 4 + \cdots n^2 + (n+1)^2$$
$$= A(n) + (n+1)^2$$
$$= B(n) + (n+1)^2$$
$$= \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6}$$
$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}$$

~~B(n+1)~~

$$= \frac{(n+1)[n(2n+1) + 6(n+1)]}{6}$$
$$= \frac{(n+1)[2n^2 + 7n + 6]}{6}$$

$$B(n+1) = \frac{(n+1)(n+2)(2n+3)}{6}$$
$$= \frac{(n+1)[2n^2 + 7n + 6]}{6}.$$

Therefore $A(n+1) = B(n+1)$: $P(n+1)$ is true 5

The method of proof by induction allows me to conclude that $P(n)$ is true for all natural numbers n.

Exercise 3:

Any postage value n, with $n \geq 8$ can be made with 3c and 5c stamps.

$$n = a_n \times 3 + b_n \times 5$$

a_n and b_n need to be integers.
a_n and b_n are "positive".

Definition: $P(n)$: $n = a_n \times 3 + b_n \times 5$
a_n, b_n are positive integers.

We prove $P(n)$ by induction.

Basis step: $P(8)$ is true?

$$8 = 3 + 5$$

$a_8 = b_8 = 1$, positive integer

Therefore $P(8)$ is true.

Inductive step: $P(n) \rightarrow P(n+1)$, is true $\quad n \geq 8$

Direct proof: We assume $P(n)$ is true.

There exists two positive integers a_n and b_n such that

$$n = a_n \times 3 + b_n \times 5$$

$$\begin{aligned}
n+1 &= a_n \times 3 + b_n \times 5 + 1 \\
&= a_n \times 3 + b_n \times 5 + 6 - 5 \\
&= a_n \times 3 + b_n \times 5 + 2 \times 3 - 5 \\
&= (a_n + 2) \times 3 + (b_n - 1) \times 5
\end{aligned}$$

Case 1: $b_n \geq 1$

$$\left.\begin{aligned}
a_{n+1} &= a_n + 2 \geq 0 \\
b_{n+1} &= b_n - 1 \geq 0
\end{aligned}\right) \text{integers.}$$

$P(n+1)$ is true.

Case 2: $b_n = 0$

$$n = 3a_n + 5b_n$$

$$n = 3a_n$$

$$n+1 = 3a_n + 1$$

$$n+1 = 3a_n + 10 - 9$$

$$n+1 = 3(a_n - 3) + 2 \times 5$$

However: $n \geq 8$

$$3a_n \geq 8 \rightarrow a_n \geq 3$$

$$\left.\begin{array}{l} a_{n+1} = a_n - 3 \geq 0 \\ b_{n+1} = 2 \geq 0 \end{array}\right) \text{integers.}$$

$P(n+1)$ is true.

The method of proof by induction allows me to conclude that $P(n)$ is true for all $n \geq 8$.