Logic

The language of mathematics

1) Objects in logic are proposition.

2) Composing propositions
→ operators

2 types of operators:

- work on 1 proposition: $\neg p$

- work on 2 propositions

or : $p \vee q$
and : $p \wedge q$
Xor : $p \oplus q$

Capture how a ~~prop~~ operator works is to define its truth table.

p	$\neg p$
T	F
F	T

p	q	$p \vee q$	$p \wedge q$	$p \oplus q$
T	T	T	T	F
T	F	T	F	T
F	T	T	F	T
F	F	F	F	F

Comparing proposition

Logical equivalence: two propositions p and q are logically equivalent if and only if they have always the same truth values.

We write $p \Leftrightarrow q$.

De Magan's law:

$$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$$

$$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$$

Definition:

- A (compound) proposition is a tautology if it is always true.
- A (compound) proposition is a contradiction if it is always false.

$A = p \vee \neg p$ is a tautology

p	$\neg p$	A
T	F	T
F	T	T

$B = p \wedge \neg p$ is a contradiction

p	$\neg p$	B
T	F	F
F	T	F

A difficult operator: the conditional

Definition Let p and q be 2 propositions. We define the compound proposition conditional, written as "$p \to q$" and meaning p implies q as the proposition that is false if p is true and q is false, and true otherwise.

Truth table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Example: If you cheat on an exam, then you get a 0.

p: you cheat on an exam

q: you get a 0.

If p then q: as ~~a~~ a contract.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

(contract)

(break the contract)

Two interesting properties of conditional

① $(p \to q) \iff \neg q \to \neg p$

② $(p \to q) \iff \neg p \vee q$

Let us prove property 1

Let us define $A = p \to q$ and $B = \neg q \to \neg p$.

p	q	$A = p \to q$	$\neg q$	$\neg p$	$B = \neg q \to \neg p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

A and B always have the same truth value

they are logically equivalent

For an implication $p \Rightarrow q$, ~~the~~

a) the propositions $\neg q \to \neg p$ is called the contrapositive of $p \Rightarrow q$, which is logically equivalent

b) the proposition $\neg p \to \neg q$ is called the converse of $p \to q$; the converse, however, is not logically equivalent.

p	q	$p \to q$	$\neg p$	$\neg q$	$\neg p \to \neg q$
T	T	T	F	F	T
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	T	T	T

Show $\underbrace{p \to q}_{A} \Leftrightarrow \underbrace{\neg p \vee q}_{B}$.

p	q	$A = p \to q$	$\neg p$	q	$B = \neg p \vee q$
T	T	T	F	T	T
T	F	F	F	F	F
F	T	T	T	T	T
F	F	T	T	F	T

Example of use:

What is $\neg(p \to q)$?

$$\neg(p \to q) \iff \neg(\neg p \vee q)$$

$$\iff \neg(\neg p) \wedge \neg q$$

$$\boxed{\neg(p \to q) \iff p \wedge \neg q}$$