Proofs 2/10/25

I) Proofs of implications

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Direct proof: To validate an implication $p \Rightarrow q$ is true, I can set p to true and prove then that q is necessarily true.

Example: Let n be an integer.

Show that if n is odd then n^2 is odd.

p : n is odd

q : n^2 is odd.

Premise:

p is true

n is odd: there exist an integer k such that $n = 2k+1$

q?

$$n^2 = (2k+1)^2$$
$$= (2k)^2 + 1^2 + 2 \times 2k \times 1$$
$$= 4k^2 + 4k + 1$$
$$= 2\underbrace{(2k^2+2k)}_{\text{integer}} + 1$$

This tells me that n^2 is odd: q is true.

Based on the concept of direct proof, I can conclude that $p \to q$ is true

Indirect proof or proof by contrapositive

We know that $p \to q$ is logically equivalent to $\neg q \to \neg p$.

To validate the implication $p \to q$, we set $\neg q$ as true, and prove that $\neg p$ is true.

Example: Let n be an integer.
Show that if n^3 is odd, then n is odd.

p : n^3 is odd $\neg p$: n^3 is even

direct proof   indirect proof

q : n is odd $\neg q$: n is even

Premise: $\neg q$ is true

n is even: there exists an integer k such that

$$n = 2k$$

$\neg p$? $\quad n^3 = (2k)^3 = 2^3 k^3 = 8k^3 = 2[\underbrace{4k^3}_{\text{integer}}]$

n^3 is even $\to$ $\neg p$ is true

Based on the concept of indirect proofs, I can conclude $p \to q$ is true.

<u>Proof by cases:</u>

Example: Let m and n be 2 integers.

Show that if mn is odd then m is odd and n is odd.

p: mn is odd $\neg p$: mn is even

q: m is odd $\wedge$ n is odd $\neg q$: m is even $\vee$ n is even.

$$\boxed{\neg(a \wedge b) \Leftrightarrow \neg a \vee \neg b}$$

Premise:

$\neg q$ is true: $\underbrace{m \text{ is even}}_{q_1} \vee \underbrace{n \text{ is even}}_{q_2}$.

I want to show $\neg q \to \neg p$

or equivalently $q_1 \vee q_2 \to \neg p$

$$\boxed{(p \vee q) \to r \iff (p \to r) \wedge (q \to r)}$$

$$\underbrace{q_1 \vee q_2}_{\neg q} \to \neg p \iff (q_1 \to \neg p) \wedge (q_2 \to \neg p)$$

2 cases:

Case 1: q_1 is true: m is even

there exists an integer k such that

$$m = 2k$$

$\neg p$: $mn = 2kn = 2\underbrace{(kn)}_{\text{integer}}$

mn is even $\neg p$ is true

Case 2: q_2 is true n is even

there exists an integer l such that

$$n = 2l$$

$$mn = m2l = 2\underbrace{(ml)}_{\text{integer}}$$

mn is even: $\neg p$ is true.

In all cases, I showed that $\neg p$ is true.

Therefore, $p \to q$ is true.

Example:

Show that if n is an integer then $n(n+1)$ is even.

p : n is an integer $\qquad \neg p$: n is not an integer

q : $n(n+1)$ is even. $\qquad \neg q$: $n(n+1)$ is odd

Direct proof:

Premise: p is true : n is an integer.

$\underbrace{n \text{ is even}}_{p_1}$ OR $\underbrace{n \text{ is odd}}_{p_2}$.

I want to prove $(p_1 \vee p_2) \to q$

logically equivalent to: $(p_1 \to q) \wedge (p_2 \to q)$

Case 1 $\quad p_1 \to q$

n is even $\to$ there exists an integer k such that $n = 2k$

$$n(n+1) = 2k(n+1) = 2[\underbrace{k(n+1)}_{\text{integer}}]$$

$n(n+1)$ is even, q is true.

Case 2 : p_2 is true

n is odd: there exists an integer k such that $n = 2k+1$

$$n(n+1) = n(2k+1+1)$$
$$= n(2k+2)$$
$$= 2(\underbrace{n(k+1)}_{\text{integer}})$$

$n(n+1)$ is even

In all cases, I have shown that $n(n+1)$ is even. $p \to q$ is true.