Proofs 2/12/25

I) Proofs associated with implications.

Let n be an integer.

1. If n is even, then n^2 is even.
2. If n^2 is even, then n is even.

p: n is even

q: n^2 is even.

1. $p \to q$
2. $q \to p$

$$(p \to q) \wedge (q \to p) \iff p \iff q$$

n is even if and only if n^2 is even.

3. $n(n+1)$ is always even.

Exercise: Let n be an integer.

Use a direct proof to show that

if n^2 is even then n is even.

p: n^2 is even

q: n is even.

Direct proof:

Premise: p is true, n^2 is even

$$n^2 = 2k \quad , \quad k \text{ integer.}$$

$$n^2 + n = 2k + n$$

$$n(n+1) = 2k + n$$

$n(n+1)$ is even: there exists an integer ℓ such that $n(n+1) = 2\ell$

$$2\ell = 2k + n$$

$$n = 2\ell - 2k = 2\underbrace{(\ell - k)}_{\text{integer}}$$

n is even.

II) Proof by contradiction.

Let p be a proposition. If we can show that $\neg p \to F$ then necessarily p is true.

p	$\neg p$	F	$\neg p \to F$
T	F	F	T
F	T	F	F

Example:

Show that $\sqrt{2}$ is irrational

p: $\sqrt{2}$ is irrational
meaning $\sqrt{2}$ is not rational.

Proof by contradiction:

I assume p is false

$\sqrt{2}$ is rational

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A number is rational if it is the ratio of 2 integers.

$$\sqrt{2} = \frac{a}{b}, \text{ with } b \neq 0$$

a and b do not have a common factor.

$$2 = \frac{a^2}{b^2}$$

$$a^2 = 2\underbrace{b^2}_{\text{integer}}$$

a^2 is even therefore $\boxed{a \text{ is even}}$

a is even: there exists an integer k such that $\underline{a=2k}$

$$a^2 = 2b^2$$

$$(2k)^2 = 2b^2$$

$$4k^2 = 2b^2$$

$$2k^2 = b^2$$

b^2 is even therefore $\boxed{b \text{ is even}}$

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Therefore the assumption that p is false leads to a contradiction (a and b are even when it was said that a and b do not have a common factor).

Therefore p is true.