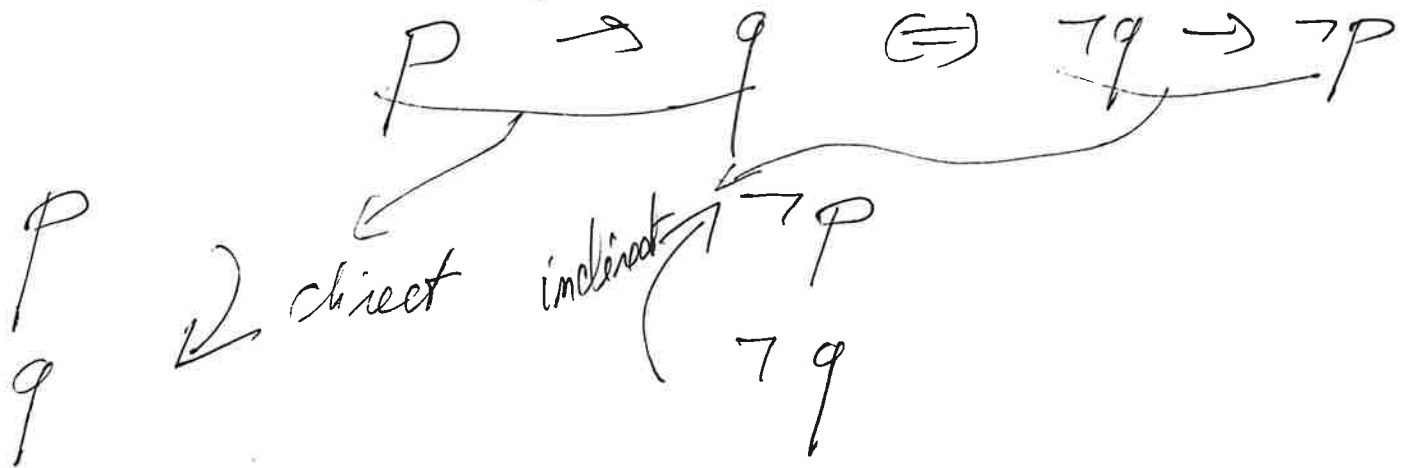


Proof (2/13/26)

①

I) Proofs of implications.



Example: let m and n be two integers. Show that

if $m > 0$ and $n \leq -2$, then $m^2 + mn + n^2 \geq 0$

$P: m > 0$ and $n \leq -2$ $\neg P: m \leq 0$ OR $n > -2$

$Q: m^2 + mn + n^2 \geq 0$ $\neg Q: m^2 + mn + n^2 < 0$

I start with a direct proof.

I assume p is true:

(2)

$$m > 0 \text{ AND } m \leq -2$$

what about q ?

$$m^2 + mn + m^2 = m^2 + 2mn + m^2 - mn$$

$$= (m+m)^2 - mn$$

$$x m \hookrightarrow m \leq -2$$

$$mn \leq -2m \leq 0$$

$$\text{then } -mn \geq 0$$

$$\text{Therefore } (m+m)^2 - mn \geq 0$$

q is true.

Conclusion : I have shown $p \rightarrow q$ using
a direct proof. 

(3).

II) Proof by contradiction

To show that a proposition p is true,
I first assume that p is false,
and I show that this leads to
a contradiction.

What if p is an implication?
 $p: a \rightarrow b$

What does it mean I assume
 p is false ($\neg p$ is true)?

$$\neg(a \rightarrow b) \Leftrightarrow \neg(\neg a \vee b) \\ \Leftrightarrow a \wedge \neg b$$

(4)

To show that a proposition
 $p \rightarrow q$ is true by contradiction,
We assume p is true AND q is false.

Example:

Let n be an integer.

Show that if n^2 is even then n is even.

p : n^2 is even
 q : n is even

$\neg p$: n^2 is odd
 $\neg q$: n is odd.

Proof by contradiction

I assume p is true AND q is false.

p is true : there exists an integer k such that $n^2 = 2k$

$\neg q$ is true : there exists an integer l such that $n = 2l + 1$

$$n^2 = 2k$$

$$(2l+1)^2 = 2k$$

$$4l^2 + 4l + 1 = 2k$$

$$2 \left[\underbrace{2p^2 + 2p}_{\text{integers}} \right] + 1 = 2k \quad (5)$$

odd even.

This is not possible. Therefore
 the assumption $(p \rightarrow q)$ is false, is false.
 $p \rightarrow q$ is true.

Conclusion: I have shown $(p \rightarrow q)$ true
 using a proof by contradiction.

Example:

$$\begin{aligned} \text{Let } a &= 2^{2026} - 5^{1024} + 7^{512} \\ b &= 3^{2025} - 6^{1023} + 8^{513} \\ c &= 4^{2027} - 8^{1025} + 9^{510} \end{aligned}$$

Show that at least one of the
 product of 2 of these numbers is positive.
 (??)

$$P = \{ab, ac, bc\}$$

⑥

P : Either ab or ac or bc
is positive.

Contradiction: Assume P is false.
 $ab < 0$ and $ac < 0$ and $bc < 0$.

The product of 3 negative numbers
is negative.

$$ab \cdot ac \cdot bc < 0.$$

$$a^2 b^2 c^2 < 0.$$

$$(abc)^2 < 0.$$

but a square cannot be negative.

Therefore P is true

(by proof by contradiction).