Proof (2/13/26)

I) Proofs of implications

$$p \to q \iff \neg q \to \neg p$$

p

q

direct

indirect

$\neg p$

$\neg q$

Example: Let m and n be two integers. Show that

if $m > 0$ and $n \leq -2$, then $m^2 + mn + n^2 \geq 0$

p: $m > 0$ and $n \leq -2$ $\quad$ $\neg p$: $m \leq 0$ OR $n > -2$

? $\quad$?

q: $m^2 + mn + n^2 \geq 0$ $\quad$ $\neg q$: $m^2 + mn + n^2 < 0$

I start with a direct proof.

I assume p is true:

$$m > 0 \text{ AND } n \leq -2$$

what about q?

$$m^2 + mn + n^2 = m^2 + 2mn + n^2 - mn$$

$$= (m+n)^2 - mn$$

$$n \leq -2$$

$\times m$

$$mn \leq -2m \leq 0$$

then $-mn \geq 0$

Therefore $(m+n)^2 - mn \geq 0$.

q is true.

<u>Conclusion</u>: I have shown $p \to q$ using a direct proof.

II) Proof by contradiction

To show that a proposition p is true, I first assume that p is false, and I show that this leads to a contradiction.

What if p is an implication?

$$p: \quad a \rightarrow b$$

What does it mean I assume p is false ($\neg p$ is true)?

$$\neg(a \rightarrow b) \Leftrightarrow \neg(\neg a \vee b)$$

$$\Leftrightarrow a \wedge \neg b$$

To show that a proposition $p \to q$ is true by contradiction,
We assume p is true AND q is false.

Example:
Let n be an integer.
Show that if n^2 is even then n is even.

p : n^2 is even
q : n is even

$\neg p$: n^2 is odd
$\neg q$: n is odd.

Proof by contradiction
I assume p is true AND q is false.
p is true : there exists an integer k such that $n^2 = 2k$
$\neg q$ is true : there exists an integer l such that $n = 2l + 1$

$$n^2 = 2k$$
$$(2l+1)^2 = 2k$$
$$4l^2 + 4l + 1 = 2k$$

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$$\underbrace{2\,[\underbrace{2p^2+2p}_{\text{integer}}]+1}_{\text{odd}} = \underbrace{2k}_{\text{even}}.$$

This is not possible. Therefore the assumption $\neg(p \to q)$ is false, is false.

$p \to q$ is true.

Conclusion: I have shown $(p \to q)$ true using a proof by contradiction.

Example:

Let $a = 2^{2026} - 5^{1024} + 7^{512}$

$b = 3^{2025} - 6^{1023} + 8^{513}$

$c = 4^{2027} - 8^{1025} + 9^{510}$

Show that at least one of the product of 2 of these numbers is positive. (≥ 0)

$P = \{ab,\ ac,\ bc\}$

P: Either ab or ac or bc is positive.

Contradiction: Assume P is false.

$ab < 0$ and $ac < 0$ and $bc < 0$.

The product of 3 negative numbers is negative.

$$ab\ ac\ bc < 0.$$

$$a^2 b^2 c^2 < 0.$$

$$(abc)^2 < 0.$$

but a square cannot be negative.

Therefore P is true

(by proof by contradiction).