Proofs 2/14

<u>Direct proof:</u>

to show $p \to q$, we assume p is true and show that necessarily q is true.

<u>Indirect proof:</u>

to show $p \to q$, we prove instead that $\neg q \to \neg p$ by assuming $\neg q$ is true and showing that $\neg p$ is true.

<u>Example:</u> Let x be a positive real number. Show that

if x is irrational, then $\sqrt{x}$ is irrational.

p: x is irrational $\qquad$ $\neg p$ x is rational

$\downarrow$? $\qquad\qquad\qquad\qquad$ $\uparrow$?

q: $\sqrt{x}$ is irrational $\qquad$ $\neg q$: $\sqrt{x}$ is rational

Indirect proof:

I assume $\neg q$ is true.

$\sqrt{x}$ is rational

There exist 2 integers a and b, with $b \neq 0$ such that

$$\sqrt{x} = \frac{a}{b}$$

square

$$x = \frac{a^2}{b^2} \begin{array}{l} \text{) integer} \\ \text{) integer, } \neq 0 \end{array}$$

therefore x is a rational number.

$\neg p$ is true.

I have shown that $\neg q \to \neg p$ is true,

therefore $p \to q$ is true.

Proof by Contradiction

Let us assume we want to prove that a proposition p is true.

Mistake : We could assume p is true, and show that it leads to something that is always true : T

p	T	$p \to T$
T	T	T
F	T	T

Correct approach We can assume that $\neg p$ is true ; we hope to show that this leads to a contradiction.

This is valid:

$\neg p$	p	F	$\neg p \to F$
F	T	F	T
T	F	F	F

$\neg p \Rightarrow F$ is only true when p is true.

This is the concept of a proof by contradiction.

Example:

p: $\sqrt{2}$ is irrational.

proof by contradiction:

I assume $\neg p$ is true.

This means that $\sqrt{2}$ is rational.

There exists 2 integers a and b, with no common factors, with $b \neq 0$

$$\sqrt{2} = \frac{a}{b}$$

$$a = \sqrt{2}\, b$$

$$a^2 = 2 \underbrace{b^2}_{\text{integer}}$$

therefore a^2 is an even number

From a result in class, we know that a is even.

There exists an integer k such that $a = 2k$

We know $a^2 = 2b^2$

$$(2k)^2 = 2b^2$$

$$4k^2 = 2b^2$$

$$2\underbrace{k^2}_{\text{integer}} = b^2$$

b^2 is even ; b is even.

If I assume $\sqrt{2} = \frac{a}{b}$, where a and b are integers, with no common factors, and $b \neq 0$ then I find a and b are even. which would mean that a and b have a common factor, 2. This is a contradiction. Therefore $\neg p$ cannot be true, p is then true.