Induction proof

I) Intuition

The blue eye problem

You are on an island with 100 others.
You could have blue or black eyes.
You cannot communicate.

Master: 1) Every 8pm, a boat comes to the island. If you tell the captain your eye color, and it is true, you may leave. If you are wrong, the captain shoots you.

2) At least one of you has blue eyes.

3) You can try on multiple days.

Assume there are 50 people with blue eyes. What happen?

We use a method that is progressive in the number of people with blue eyes.

The blue eye problem

$P(n)$: If there are n people with blue eyes, those people will leave the island on the n-th day.

$P(1)$ is true.

$P(2)$ is also true.

$P(3)$ is also true.

$\vdots$

$P(n)$ is also true.

Let us assume that

$P(n)$ is true.

What about $P(n+1)$?

If you have $n+1$ people with blue eyes:

- no one would have left before day $n+1$.

During day $(n+1)$:

One person with blue eyes: B

B sees n people with blue eyes. If they were exactly n people with blue eyes, they would have left on day n. B knows he has blue eyes... all the people with blue eyes $(n+1)$ will leave on the $(n+1)$th day: $P(n+1)$ is true.

<u>Theorem</u>: Let P be a property that depends on a natural number n. If I can show that

- $P(1)$ is true
- $P(n) \rightarrow P(n+1)$ is true, when $n \geq 1$

then $P(n)$ is true for all natural numbers. This concept is the method of proof by induction.

$$\left[P(1) \wedge \left[P(n) \rightarrow P(n+1) \right] \right] \rightarrow P(n), \forall n$$

Example:

Show that

$$1 + 2 + \cdots\cdots n = \frac{n(n+1)}{2}$$

for all natural number n.

① Define the problem

$LHS(n) = 1 + 2 + \cdots\cdots n$

$RHS(n) = \frac{n(n+1)}{2}$

$P(n): \quad LHS(n) = RHS(n)$

I want to show $P(n)$ is true for all n.

② Method of proof

Proof by induction:

(Basis step) • $P(1)$ is true

(Inductive step) • $P(n) \rightarrow P(n+1)$ is true, for all $n \geq 1$

3 <u>Basis step</u> : $P(1)$ is true.

$$LHS(1) = 1$$
$$RHS(1) = \frac{1 \times (1+1)}{2} = 1$$
$> P(1)$ is true.

4 <u>Inductive step</u>

$$P(n) \to P(n+1) \qquad n \geq 1.$$

<u>Premise</u> $P(n)$ is true,

$$LHS(n) = RHS(n)$$

$$LHS(n+1) = \underbrace{1 + \cdots + n} + n + 1$$
$$= LHS(n) + n + 1$$
$$= RHS(n) + n + 1$$
$$= \frac{n(n+1)}{2} + n + 1$$
$$= \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

$$RHS(n+1) = \frac{(n+1)(n+2)}{2}$$

$P(n+1)$ is true.

5 Conclusion

The method of proofs by induction allows me to conclude that $P(n)$ is true for all natural number n.

Anecdote:

$$1 + \cdots + n = \frac{n(n+1)}{2}$$

$$S = 1 + 2 \cdots + n$$
$$S = n + n-1 + \cdots 1$$

$$2S = (n+1) + (n+1) + \cdots (n+1)$$

$$2S = n(n+1)$$

$$\rightarrow \boxed{S = \frac{n(n+1)}{2}}$$