The blue eye problem

$P(n)$: If there are n people with blue eyes, those people will leave the island on the n-th day.

$P(1)$ is true.

$P(2)$ is also true.

$P(3)$ is also true.

$\vdots$

$P(n)$ is also true.

Let us assume that

$P(n)$ is true.

What about $P(n+1)$?

If you have $n+1$ people with blue eyes:

- no one would have left before day $n+1$.

During day $(n+1)$:

One person with blue eyes: B

B sees n people with blue eyes.

if they were exactly n people with blue eyes, they would have left on day n.

B knows he has blue eyes ... all the people with blue eyes $(n+1)$ will leave on the $(n+1)$th day: $P(n+1)$ is true.

<u>Theorem</u>: Let P be a property that depends on a natural number n.
If I can show that

- $P(1)$ is true
- $P(n) \rightarrow P(n+1)$ is true, when $n \geq 1$

then $P(n)$ is true for all natural numbers.

This concept is the method of proof by induction.

$$\left[P(1) \wedge \left[P(n) \rightarrow P(n+1) \right] \right] \rightarrow P(n), \forall n$$

Example:

Show that

$$1 + 2 + \cdots n = \frac{n(n+1)}{2}$$

for all natural number n.

① Define the problem

$$LHS(n) = 1 + 2 + \cdots n$$

$$RHS(n) = \frac{n(n+1)}{2}$$

$$P(n): \quad LHS(n) = RHS(n)$$

I want to show $P(n)$ is true for all n.

② Method of proof

Proof by induction:

(Basis step) • $P(1)$ is true

(Inductive step) • $P(n) \to P(n+1)$ is true, for all $n \geq 1$

3 <u>Basis step</u> : $P(1)$ is true.

$LHS(1) = 1$

$RHS(1) = \frac{1 \times (1+1)}{2} = 1$

$>$ $P(1)$ is true.

4 <u>Inductive step</u>

$P(n) \rightarrow P(n+1) \quad n \geq 1$.

<u>Premise</u> $P(n)$ is true,

$LHS(n) = RHS(n)$

$$LHS(n+1) = \underbrace{1 + \cdots + n}_{} + n + 1$$

$$= LHS(n) + n + 1$$

$$= RHS(n) + n + 1$$

$$= \frac{n(n+1)}{2} + n + 1$$

$$= \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

$$RHS(n+1) = \frac{(n+1)(n+2)}{2}$$

$P(n+1)$ is true.

5 Conclusion

The method of proofs by induction allows me to conclude that $P(n)$ is true for all natural number n.

Anecdote:

$$1 + \cdots + n = \frac{n(n+1)}{2}$$

$$S = 1 + 2 \cdots + n$$
$$S = n + n-1 + \cdots 1$$

$$2S = (n+1) + (n+1) + \cdots (n+1)$$

$$2S = n(n+1)$$

$$\rightarrow \boxed{S = \frac{n(n+1)}{2}}$$

Beware !

(False proof):

All bunnies have the same color

Let us rewrite:

n bunnies have the same color.

1. Defining the problem:

$P(n)$: n bunnies have the same color.

I want to prove $P(n)$ is true for all n.

2. Method of proof:

Proof by induction:

- Basis step: $P(1)$ is true
- Inductive step: $P(n) \rightarrow P(n+1)$

3) Basis step:

$P(1)$: 1 bunny has the same color as itself.

$P(1)$ is true.

4) Inductive step: $P(n) \to P(n+1)$

Premise: $P(n)$ is true. Whenever you take n bunnies, they all have the same color.

$P(n+1)$:

$n+1$ bunnies: $B_1, \ldots\ldots, B_{n+1}$

Group 1: $\{B_1, \ldots, B_n\} \to C_1$

n bunnies.

does not exist if $n = 1$

$\{B_2 \ldots\ldots, B_{n+1}\} \to C_2$

All $\{B_2, \ldots\ldots B_n\}$ have color C_1 and C_2

Therefore $C_1 = C_2 = C$.