Induction 2/26

I) Proof by induction

Let us assume that we want to prove that

$$P(n) \text{ is true, } \forall n \geq n_0$$

A proof by induction involves two steps:

<u>Basis step</u>: We show $P(n_0)$ is true

<u>Inductive step</u>: We prove $p(n) \to p(n+1)$ is true for all $n \geq n_0$

The method of proof by induction allows us to conclude that $P(n)$ is true for all $n \geq n_0$.

II) Examples

II) 1). Let n be a natural number

Show that $\sum_{i=1}^{n} (2i-1) = n^2$

$$1 + 3 + 5 + \dots 2n-1 = n^2$$

We use a proof by induction:

Let us define:

$$A(n) = \sum_{i=1}^{n} (2i-1) = 1 + 3 + \dots 2n-1$$

$$B(n) = n^2$$

$$P(n): \quad A(n) = B(n)$$

Basis step: I prove $P(1)$ is true

$$\left.\begin{array}{l} A(1) = 1 \\ B(1) = 1^2 = 1 \end{array}\right\} P(1) \text{ is true}$$

Inductive step: $P(n) \to P(n+1) \quad \forall n \geq 1$

Let us assume $P(n)$ is true.

$$A(n) = B(n)$$

$$A(n+1) = \sum_{i=1}^{n+1} 2i-1 = \underbrace{1+3+\dots+2n-1}+2n+1$$

$$A(n+1) = A(n) + 2n+1$$

$$= B(n) + 2n+1$$

$$= n^2 + 2n+1$$

$$= (n+1)^2$$

$$B(n+1) = (n+1)^2$$

therefore $P(n+1)$ is true.

The method of proof by induction allows me to conclude that $P(n)$ is true for all natural numbers n.

Example 2

Let n be a natural number.

Can you find a simpler expression for $S(n) = \sum_{i=1}^{n} \frac{1}{2^i}$?

$$S(1) = \frac{1}{2} \qquad \frac{2-1}{2}$$

$$S(2) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = \frac{2^2-1}{2^2}$$

$$S(3) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} = \frac{2^3-1}{2^3}$$

$$S(4) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16} = \frac{2^4-1}{2^4}$$

Intuition:

$$S(n) = \frac{2^n - 1}{2^n}$$

We prove this intuition by induction.

Definitions:

$$S(n) = \sum_{i=1}^{n} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{2^2} + \cdots \frac{1}{2^n}$$

$$B(n) = \frac{2^n - 1}{2^n}$$

$$P(n): \quad S(n) = B(n)$$

Proof by induction

Basis step: $P(1)$ is true

$$\left.\begin{aligned} S(1) &= \frac{1}{2} \\ B(1) &= \frac{2-1}{2} = \frac{1}{2} \end{aligned}\right\} P(1) \text{ is true}$$

Inductive step: $P(n) \to P(n+1)$ is true, $\forall n \geq 1$.

Assume $P(n)$ is true: $S(n) = B(n)$

$$S(n+1) = \underbrace{\frac{1}{2} + \cdots \frac{1}{2^n}} + \frac{1}{2^{n+1}}$$

$$S(n+1) = S(n) + \frac{1}{2^{n+1}}$$

$$= \frac{2^n - 1}{2^n} + \frac{1}{2^{n+1}} = \frac{2(2^n - 1) + 1}{2^{n+1}}$$

⑥

$$S(n+1) = \frac{2(2^n-1)+1}{2^{n+1}}$$

$$= \frac{2^{n+1}-2+1}{2^{n+1}} = \frac{2^{n+1}-1}{2^{n+1}}$$

$$B(n+1) = \frac{2^{n+1}-1}{2^{n+1}}$$

Therefore $S(n+1) = B(n+1)$, $P(n+1)$ is true.

The method of proof by induction allows me to conclude that

$$\sum_{i=1}^{n} \frac{1}{2^i} = \frac{2^n-1}{2^n},$$ for all natural numbers n.

Danger !!

I will prove that
all bunnies have the same color.

$P(n)$: In a group of n bunnies, these bunnies have the same color.

We prove that $P(n)$ is true for all n, natural number.

Basis step : $P(1)$?
One bunny has one color: $P(1)$ is true.

Inductive step $P(n) \to P(n+1)$ $\forall n$

I assume $P(n)$ is true.
If you have a group of n bunnies, these n bunnies have the same color.

$P(n+1)$?

$G = \{ B_1, B_2, \ldots, B_n, B_{n+1} \}$

$$G_1 = \{ \underset{\downarrow}{B_1}, \underset{\downarrow}{B_2} \ldots \underset{\downarrow}{B_n} \}$$
$$\quad\;\; C_1 \quad C_1 \quad\;\; C_1$$

$$G_2 = \{ \underset{\downarrow}{B_2}, \ldots, \underset{\downarrow}{B_n}, \underset{\downarrow}{B_{n+1}} \}$$
$$\quad\;\; C_2, \ldots, C_2, C_2$$

$G = \underline{\{ B_2, \ldots B_n \}}$

those bunnies have both the color C_1 and C_2 : therefore $C_1 = C_2$

$P(n+1)$ is true.

The method of proof by induction allows me to conclude that all bunnies have the same color.