Proofs 2/3/25

I) Theorems

Definition: A theorem is a proposition that can be shown to be always true.

We demonstrate that a theorem is true with a sequence of propositions that form an argument, also called a proof.

Proof includes:

Premises

Body of the proof. { existing knowledge, rules of inference.

Conclusion

II Rule of inference: "modus ponens"

if $p \to q$ is true

and p is true

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then q is true

Why this is true?

p	q	$p \to q$	
T	T	T	
T	F	F	as $p \to q$ is true.
F	T	T	
F	F	T	/ as p is true.

Inspector Craig from Scotland Yard.

Shipments of expensive items have been going missing from the docks. The notorious criminals A, B, and C are brought in for questioning. The following evidences were established:

- B is guilty
- if B is guilty, then C is innocent.
- if C is innocent, then A is guilty.
- At least one of A, B, or C is guilty.

Express it in the language of mathematics.

A: A is guilty

B: B is guilty

C: C is guilty.

Evidence:

B is true

$B \rightarrow \neg C$ is true

$\neg C \rightarrow A$ is true

$A \vee B \vee C$ is true.

B is true (1)

$B \rightarrow \neg C$ is true (2)

$\neg C \rightarrow A$ is true (3)

$A \vee B \vee C$ is true (4)

Use (1) and (2):

$$\frac{\begin{array}{l} B \text{ is true} \\ B \rightarrow \neg C \text{ is true} \end{array}}{\neg C \text{ is true}}$$

: C is innocent.

(3):
$$\frac{\begin{array}{l} \neg C \text{ is true} \\ \neg C \rightarrow A \text{ is true} \end{array}}{A \text{ is true}}$$

: A is guilty. √ √

Conclusion: A and B are guilty √ √ while C is innocent.

Mr Mc Grega calls Scotland Yard to report a robbery. The notorious criminals A, B, and C are brought in for questioning. The following evidences were established.

- B is innocent.
- if A is innocent, then B is guilty
- if C is innocent, then A is innocent.
- At least one of them is guilty.

A: A is guilty
B: B is guilty
C: C is guilty.

(1) $\neg B$ is true

(2) $\neg A \rightarrow B$ is true

(3) $\neg C \rightarrow \neg A$ is true

(4) $A \vee B \vee C$ is true.

Let us look at (1) and (2):

(1) $\neg B$ is true

(2) $\neg A \to B$ is true.

$$\boxed{P \to q \iff \neg q \to \neg P}$$

(2) $$\boxed{\neg A \to B \iff \neg B \to A}$$

(1) $\neg B$ is true

(2) $\neg B \to A$ is true

A is true: A is guilty. ✓✓

A is guilty ✓

(3) $\neg C \to \neg A$ is true.

(3) is logically equivalent to $A \to C$ is true. ✓✓

A is true

$A \to C$ is true

C is true: C is guilty ✓

Conclusion: A and C are guilty, while B is innocent.