Proofs 2/9/26

I) Direct proof

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

To prove that the proposition $p \to q$ is true, we assume p is true and show that q is necessarily true.

Let n be an integer.
Show that if n is odd, then n^2 is odd.

p: n is odd

q: n^2 is odd.

Direct proof: p is true, n is odd.

n is odd: there exists an integer k such that

$$n = 2k+1$$

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2\,\underbrace{[2k^2 + 2k]}_{\text{integer}} + 1 \end{aligned}$$

Therefore n^2 is odd: q is true.

<u>Example 2:</u> Let n be an integer.

Show that if $9n$ is even, then n is even.

p: $9n$ is even

q: n is even.

I use a direct proof.

I assume p is true, $9n$ is even.

There exists an integer k such that

$$9n = 2k.$$

$$9n = n + 8n$$

$$n + 8n = 2k$$

$$n = 2k - 8n$$

$$n = 2[\underbrace{k - 4n}_{\text{integer}}].$$

Therefore n is even. $\therefore q$ is true.

II) Proof by contrapositive

Let n be an integer.
Show that if n^2 is even, then n is even.

p: n^2 is even

q: n is even.

I try a direct proof:

I assume p is true, meaning n^2 is even.
The exists an integer k such that

$$n^2 = 2k$$

This is hard as I am not allowed to use square root.

In logic, I have seen that

$$\boxed{p \to q \iff \neg q \to \neg p}$$

p: n^2 is even $\quad\quad$ $\neg p$: n^2 is odd.

q: n is even $\quad\quad$ $\neg q$ n is odd.

Instead of showing $p \to q$, (if n^2 is even, then n is even)

I show $\neg q \to \neg p$. (if n is odd, then n^2 is odd)

III Proof by case

Example:

Show that if n is an integer, then $n(n+1)$ is even.

p: n is an integer $\quad\quad$ ~~$\neg p$: n is not an integer~~

q: $n(n+1)$ is even. $\quad\quad$ ~~$\neg q$: $n(n+1)$ is not even.~~

So, direct proof.

I assume p is true.

n is an integer.

n is an integer:

n is even or n is odd.

$p \Leftrightarrow p_1 \vee p_2$ where $\left| \begin{array}{l} p_1: n \text{ is even} \\ p_2: n \text{ is odd.} \end{array} \right.$

Therefore, I am trying to prove

$$(p_1 \vee p_2) \rightarrow q$$

However:

$$(p_1 \vee p_2) \rightarrow q \Leftrightarrow (p_1 \rightarrow q) \wedge (p_2 \rightarrow q)$$

I can use a proof by case:

<u>Case 1</u>: $p_1 \rightarrow q$

p_1: n is even

q: $n(n+1)$ is even.

Direct proof: n is even: there exists an integer k such that $n = 2k$

$$n(n+1) = 2k(2k+1) = 2\Big[\underbrace{k(2k+1)}_{\text{integer}}\Big]$$

$n(n+1)$ is even: q is true.

Case 2 : $p_2 \to q$

p_2 : n is odd

q : $n(n+1)$ is even.

Direct proof: there exists an integer k such that $n = 2k+1$

$$
\begin{aligned}
n(n+1) &= (2k+1)(2k+1+1) \\
&= (2k+1)(2k+2) \\
&= 2\big[\underbrace{(2k+1)(k+1)}_{\text{integer}}\big]
\end{aligned}
$$

Therefore $n(n+1)$ is even ; q is true.

Recall: Let n be an integer, show that if n^2 is even, then n is even.

p : n^2 is even

q : n is even.

Direct proof: n^2 is even

I assume n^2 is even.

There exists an integer k such that

$$n^2 = 2k$$

$$n^2 + n = 2k + n$$

$$n(n+1) = 2k + n.$$

There exists an integer l such that $n(n+1) = 2l$

$$2l = 2k + n$$

$$n = 2l - 2k = 2\underbrace{(l-k)}_{\text{integer}}$$

n is even.