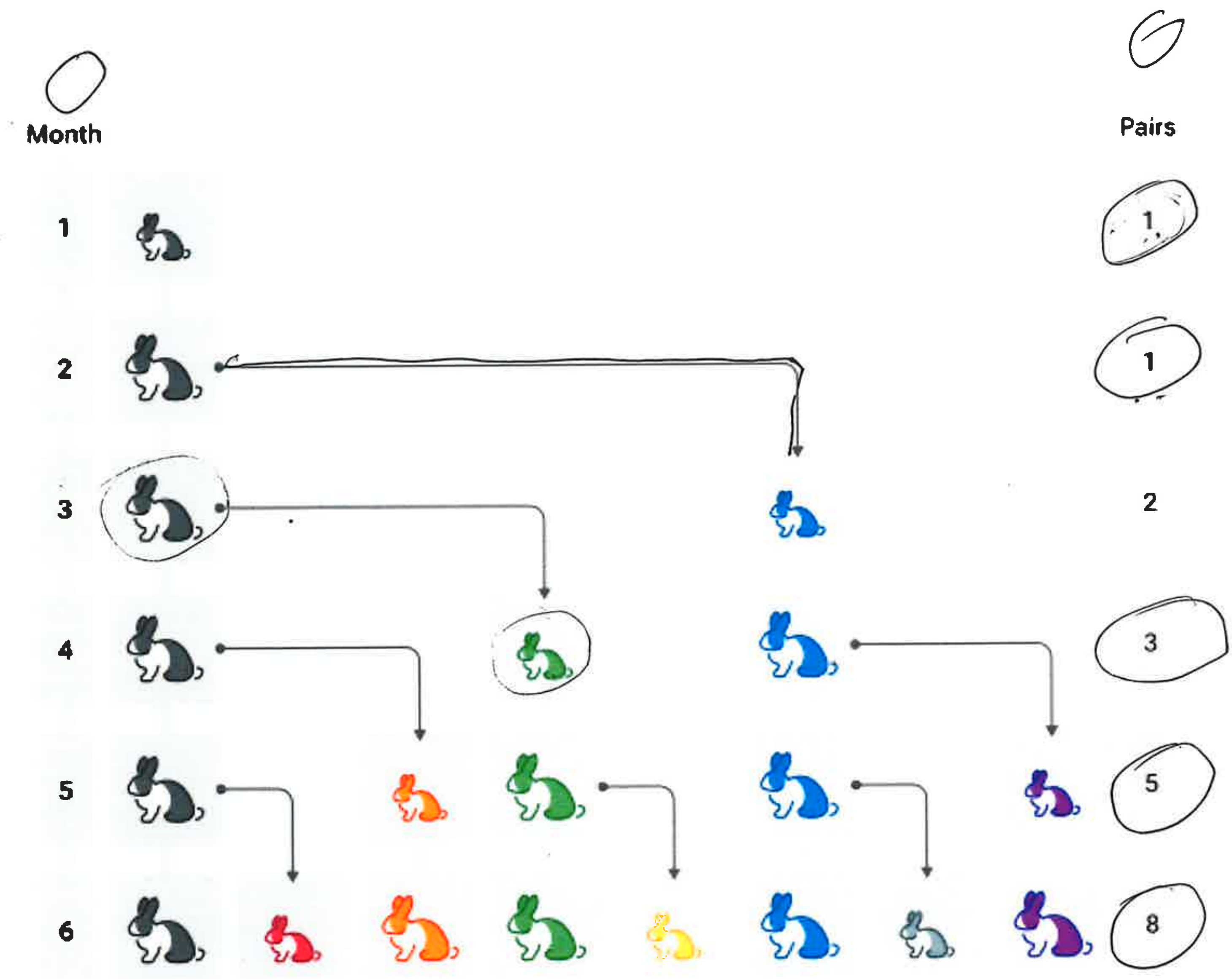
Fibonacci numbers



$$\boxed{f_n = f_{n-1} + f_{n-2}}$$

Fibonacci 3/28

1) Fibonacci numbers

Fibonacci numbers are defined as:

$$\begin{cases} F_0 = 0 \\ F_1 = 1 \end{cases}$$

$$F_n = F_{n-1} + F_{n-2} \quad n \geq 2$$

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$$

(true, but not useful).

Problem 1

Let F_n be the Fibonacci numbers.
Show that

$$F_1 + F_3 + F_5 + \dots F_{2n-1} = F_{2n}$$

for all $n \geq 1$.

Definition of the problem:

$$A(n) = F_1 + \dots\dots F_{2n-1}$$

$$B(n) = F_{2n}$$

$P(n)$: $A(n) = B(n)$

I want to show $P(n)$ is true, $\forall n \geq 1$

Method of proof: induction

Body of the proof:

a) Basis step: I want to show $P(1)$ is true.

$$A(1) = F_1 = 1$$

$$B(1) = F_2 = 1$$

$>$ $P(1)$ is true.

b) Inductive step:

$$P(n) \longrightarrow P(n+1) \qquad n \geq 1$$

Premise: $P(n)$ is true, meaning $A(n) = B(n)$ ✓

$$A(n+1) = \underbrace{F_1 + F_3 + \cdots F_{2n-1}} + F_{2n+1}$$

$$= A(n) + F_{2n+1}$$

$$= B(n) + F_{2n+1}$$

$$= F_{2n} + F_{2n+1}$$

$2n \quad 2n+1 \quad 2n+2$

$$F_{2n} + F_{2n+1} = F_{2n+2}$$

$$A(n+1) = F_{2n+2}$$

$>$ $P(n+1)$ is true.

$$B(n+1) = F_{2(n+1)} = F_{2n+2}$$

The method of proof by induction allows me to conclude $P(n)$ is true $\forall n \geq 1$.

Problem 2:

Let F_n be the Fibonacci numbers. Show that for all $n \geq 1$

$$F_1^2 + F_2^2 + \cdots F_n^2 = F_n F_{n+1}$$

Definition of the problem:

Let us define

$$A(n) = F_1^2 + \cdots F_n^2$$

$$B(n) = F_n F_{n+1}$$

$$P(n): \quad A(n) = B(n)$$

I want to show $P(n)$ is true $\forall n \geq 1$

Body of the proof:

Basis step: is $P(1)$ true?

$$A(1) = F_1^2 = 1^2 = 1$$

$$B(1) = F_1 F_{1+1} = F_1 F_2 = 1 \times 1 = 1$$

$$A(1) = B(1) \rightarrow P(1) \text{ is true.}$$

Inductive step:

$$P(n) \to P(n+1) \quad n \geq 1$$

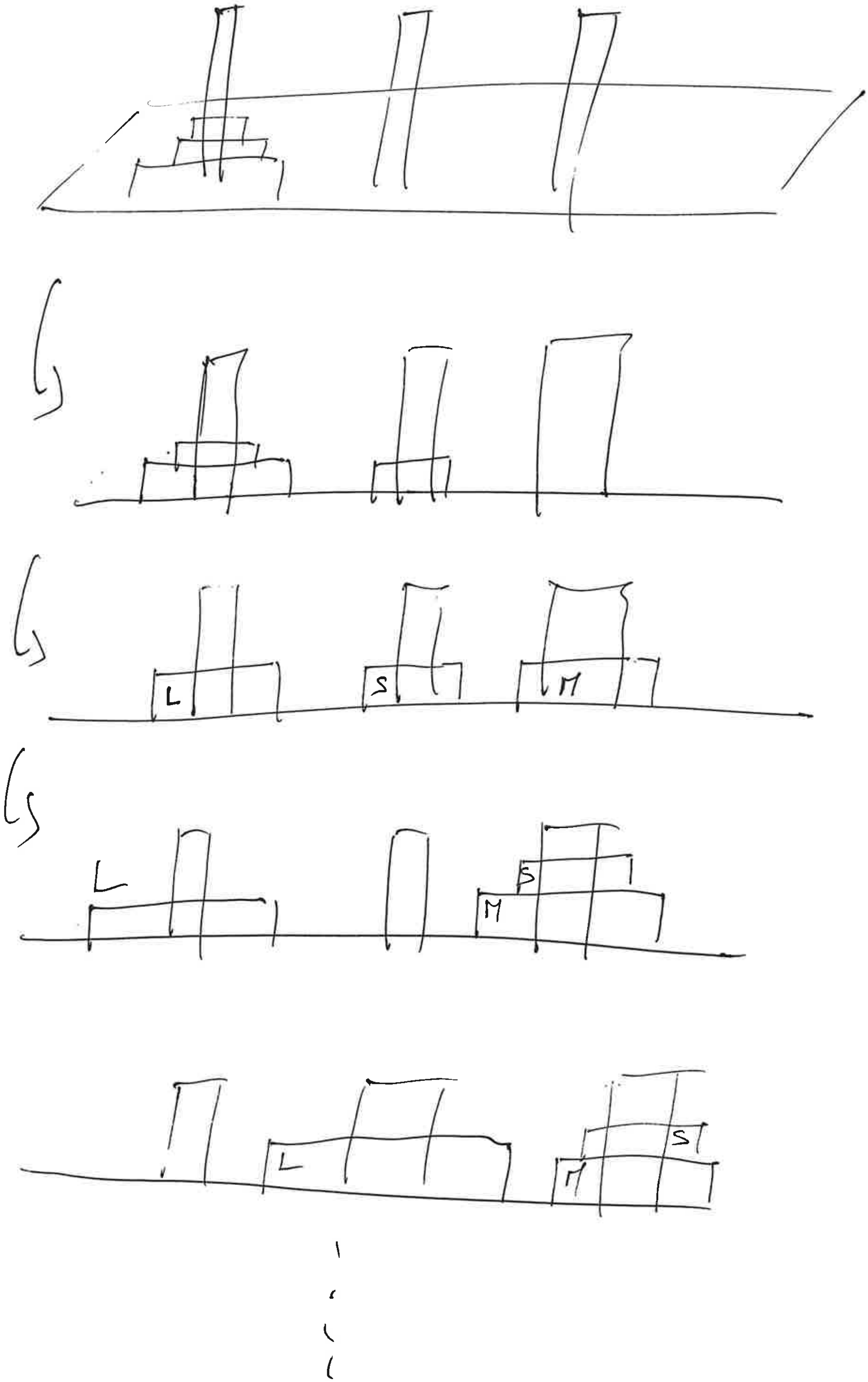
Premise : $P(n)$ is true : $A(n) = B(n)$

$$\begin{aligned} A(n+1) &= F_1^2 + \cdots F_n^2 + F_{n+1}^2 \\ &= A(n) + F_{n+1}^2 \\ &= B(n) + F_{n+1}^2 \\ &= F_n F_{n+1} + F_{n+1}^2 \\ &= (F_n + F_{n+1}) F_{n+1} \\ \cancel{B(n+1)} &= F_{n+2} F_{n+1} \end{aligned}$$

$$B(n+1) = F_{n+1} F_{n+2}$$

$$A(n+1) = B(n+1) \to P(n+1) \text{ is true.}$$

The method of proof by induction allows me to conclude that $P(n)$ is true $\forall n \geq 1$.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$P(n)$: In the tower of Hanoi game, The minimum number of moves required to move n disks from one pole to another is $2^n - 1$

Problem:

$P(n)$: $T_m(n) = 2^n - 1$.

$P(n)$ is true $n \geq 1$

Proof by induction:

Basis step $P(1)$?

$T_m(1) = 1$

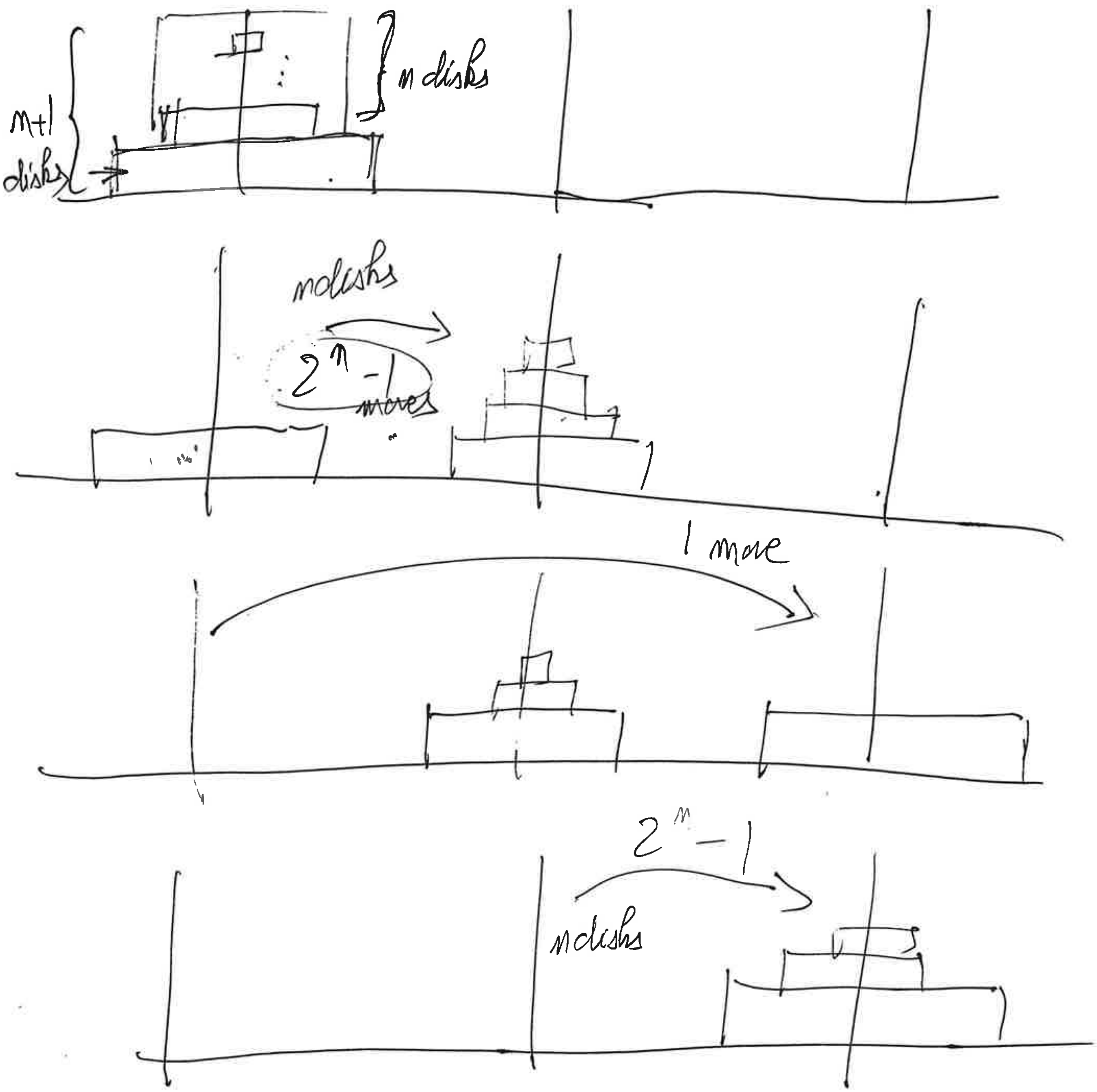
$2^n - 1 = 2 - 1 = 1$ $\Rightarrow P(1)$ is true.

Inductive step: $P(n) \rightarrow P(n+1) \quad n \geq 1$

Premise: $P(n)$ is true.

To move n disks, I need $T(n) = 2^n - 1$ moves.

$P(n+1)$?



Total number of moves:

$$T_n(n+1) = 2^n - 1 + 1 + 2^n - 1$$
$$= 2 \times 2^n - 1$$
$$= 2^{n+1} - 1 \quad P(n+1) \text{ is true.}$$

The method of proof by induction allows me to conclude that we can solve the Tower of Hanoi problem with n disks in $2^n - 1$ moves.