Discussion 8: Solutions

ECS 20 (Winter 2019)

Patrice Koehl
koehl@cs.ucdavis.edu

February 26, 2019

Induction

Exercise a

Let $P(n)$ be the proposition:

$$\sum_{i=1}^{n} (-1)^i i^2 = \frac{(-1)^n n(n+1)}{2}$$

We want to show that $P(n)$ is true for all $n > 0$. Let us define: $LHS(n) = \sum_{i=1}^{n} (-1)^i i^2$ and $RHS(n) = \frac{(-1)^n n(n+1)}{2}$.

- *Basic step*:

$$LHS(1) = (-1) \times 1^2 = 1 \qquad RHS(1) = \frac{(-1) \times 1 \times 2}{2} = 1$$

  Therefore $P(1)$ is true.

- *Induction step*: We suppose that $P(k)$ is true, with $1 \leq k$. We want to show that $P(k+1)$ is true.

$$
\begin{aligned}
LHS(k+1) &= \sum_{i=1}^{k+1}(-1)^i i^2 \\
&= \sum_{i=1}^{k}(-1)^i i^2 + (-1)^{k+1}(k+1)^2 \\
&= LHS(k) + (-1)^{k+1}(k+1)^2 \\
&= RHS(k) + (-1)^{k+1}(k+1)^2 \\
&= \frac{(-1)^k k(k+1)}{2} + (-1)^{k+1}(k+1)^2 \\
&= \frac{(-1)^k k(k+1) + 2(-1)^{k+1}(k+1)^2}{2} \\
&= \frac{(-1)^{k+1}(k+1)(2k+2-k)}{2} \\
&= \frac{(-1)^{k+1}(k+1)(k+2)}{2}
\end{aligned}
$$

and

$$RHS(k+1) = \frac{(-1)^{k+1}(k+1)(k+2)}{2}$$

Therefore $LHS(k+1) = RHS(k+1)$, which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n > 0$.

Exercise b

Let $P(n)$ be the proposition: $2^n \leq n!$. Let us define $LHS(n) = 2^n$ and $RHS(n) = n!$. We want to show that $P(n)$ is true for all $n \geq 4$.

- *Basis step:* We show that $P(4)$ is true:

  $$LHS(4) = 2^4 = 16$$
  $$RHS(4) = 4! = 24$$

  Therefore $LHS(4) \leq RHS(4)$ and $P(4)$ is true.

- *Inductive step:* Let k be a positive integer greater or equal to 4 ($k \geq 4$), and let us suppose that $P(k)$ is true. We want to show that $P(k+1)$ is true.

  $$LHS(k+1) = 2^{k+1} = 2LHS(k)$$

  Since P(k) is true, we find:

  $$LHS(k+1) \leq 2k!$$

Since $k \geq 4$, $2 \leq k+1$.
Therefore

$$\begin{aligned} LHS(k+1) &\leq (k+1) \times k! \\ LHS(k+1) &\leq (k+1)! \end{aligned}$$

Since $RHS(k+1) = (k+1)!$, we get $LHS(k+1) < RHS(k+1)$ which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n \geq 4$.

Exercise c

Let $P(n)$ be the proposition:

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

We want to show that $P(n)$ is true for all $n > 0$. Let us define: $LHS(n) = \displaystyle\sum_{i=1}^{n} \frac{1}{(i)(i+1)}$ and $RHS(n) = \frac{n}{n+1}$.

- *Basic step*:

$$LHS(1) = \frac{1}{1 \times 2} = \frac{1}{2} \qquad RHS(1) = \frac{1}{2}$$

  Therefore $P(1)$, $P(2)$ and $P(3)$ are true.

- *Induction step*: We suppose that $P(k)$ is true, with $1 \leq k$. We want to show that $P(k+1)$ is true.

$$
\begin{aligned}
LHS(k+1) &= \sum_{i=1}^{k+1} \frac{1}{i(i+1)} \\
&= \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} \\
&= LHS(k) + \frac{1}{(k+1)(k+2)} \\
&= RHS(k) + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2}
\end{aligned}
$$

and

$$
RHS(k+1) = \frac{k+1}{k+2}
$$

Therefore $LHS(k+1) = RHS(k+1)$, which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all n.

Fibonacci

Exercise a

Let $P(n)$ be the proposition: $f_1 + f_2 + \ldots + f_n = f_{n+2} - 1$. We define $LHS(n) = f_1 + f_2 + \ldots + f_n$ and $RHS(n) = f_{n+2} - 1$. We want to show that $P(n)$ is true for all n.

- *Basic step*:

$$
\begin{aligned}
LHS(1) &= f_1 = 1 \\
RHS(1) &= f_3 - 1 = 2 - 1 = 1
\end{aligned}
$$

  Therefore $LHS(1) = RHS(1)$ and $P(1)$ is true.

- *Inductive step:* Let k be a positive integer, and let us suppose that $P(k)$ is true. We want to show that $P(k+1)$ is true.

Then

$$\begin{aligned} LHS(k+1) &= f_1 + f_2 + \ldots + f_k + f_{k+1} \\ &= LHS(k) + f_{k+1} \\ &= RHS(k) + f_{k+1} \\ &= f_{k+2} - 1 + f_{k+1} \\ &= f_{k+1} + f_{k+2} - 1 \\ &= f_{k+3} - 1 \end{aligned}$$

and

$$RHS(k+1) = f_{k+3} - 1$$

Therefore $LHS(k+1) = RHS(k+1)$, which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all n.

Exercise b

Let $P(n)$ be the proposition: f_{4n} is divisible by 3. We define $LHS(n) = f_{4n}$. We want to show that $P(n)$ is true for all n.

- *Basic step*:

$$LHS(1) = f_4 = 3$$

  Therefore $LHS(1)$ is divisible by 3 and $P(1)$ is true.

- *Inductive step:* Let k be a positive integer, and let us suppose that $P(k)$ is true. The there exist m such that $LHS(k) = f_{4k} = 3m$. We want to show that $P(k+1)$ is true.
  Then

$$\begin{aligned} LHS(k+1) &= f_{4k+4} \\ &= f_{4k+3} + f_{4k+2} \\ &= 2f_{4k+2} + f_{4k+1} \\ &= 2(f_{4k+1} + f_{4k}) + f_{4k+1} \\ &= 3f_{4k+1} + 2f_{4k} \\ &= 3f_{4k+1} + 6m \\ &= 3(f_{4k+1} + 2m) \end{aligned}$$

  Therefore $LHS(k+1)$ is divisible by 3, which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all n.

Others

Exercise a

Show that $21/(4^{n+1} + 5^{2n-1})$ for all $n > 0$.

Let $P(n)$ be the proposition: $(4^{n+1} + 5^{2n-1})$ is divisible by 21. We define $A(n) = 4^{n+1} + 5^{2n-1}$. We want to show that $P(n)$ is true for all n.

- *Basis step*:

  $$A(1) = 4^2 + 5 = 16 + 5 = 21$$

  Therefore $A(1)$ is divisible by 21 and $P(1)$ is true.

  $$A(2) = 4^3 + 5^3 = 64 + 125 = 189 = 9 \times 21$$

  Therefore $A(2)$ is divisible by 21 and $P(2)$ is true.

- *Inductive step:* Let k be a positive integer, and let us suppose that $P(k)$ is true. Then there exist m such that $A(k) = 21m$, namely $4^{k+1} + 5^{2k-1} = 21m$. We want to show that $P(k+1)$ is true.
  Then

  $$\begin{aligned} A(k+1) &= 4^{k+2} + 5^{2k+1} \\ &= 4 \times 4^{k+1} + 25 \times 5^{2k-1} \\ &= 4 \times (21m - 5^{2k-1}) + 25 \times 5^{2k-1} \\ &= 21 \times (4m) + (25 - 4) \times 5^{2k-1} \\ &= 21 \times (4m) + 21 \times 5^{2k-1} \\ &= 21 \times (4m + 5^{2k-1} \end{aligned}$$

  Therefore $A(k+1)$ is divisible by 21, which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all n.

Exercise b

Show that any postage value of n cents can be composed with a combination of 4-cent and 7-cent stamps only, when n is greater or equal to 18.

Let $P(n)$ be the proposition: n cents can be composed with a combination of 4-cent and 7-cent stamps only.
We want to show that $P(n)$ is true for all $n \geq 18$.

We note first that $P(n)$ can be rewritten as: There exits a pair of integers (a, b) such that $n = 4a+7b$, with $a \geq 0$ and $b \geq 0$.
We use a proof by induction:

- *Basis step*:
  Let $n = 18$; we note that $18 = 4 + 2 \times 7$; therefore $P(18)$ is true
  Let $n = 19$; we note that $19 = 3 \times 4 + 7$; therefore $P(19)$ is true

- *Inductive step:* Let k be a positive integer; we want to show that $P(k) \to P(k+1)$ for all $k \geq 18$.
  To prove this implication, we suppose that $P(k)$ is true. Then there exist $(a, b) \in \mathbb{Z}^2$ such that $k = 4a + 7b$, with $a \geq 0$ and $b \geq 0$.
  We want to find a similar decomposition of $k+1$, namely we would like to write $k+1 = 4c+7d$, with $c \geq 0$ and $d \geq 0$. Since $k = 4a + 7b$, we have,

$$k + 1 = 4a + 7b + 1$$

  We note that $1 = 8 - 7 = 2 \times 4 - 7$. Therefore,

$$k = 4a + 7b + 2 \times 4 - 7 = 4(a + 2) + 7(b - 1)$$

  Since $a \geq 0$, $a + 2 \geq 0$. However, $b - 1 \geq 0$ if and only if $b \geq 1$. We therefore distinguish two cases:

$b \geq 1$.
Let us define $c = a+2$ and $d = b-1$. Both c and d are positive (or 0), and $k+1 = 4c+7d$. Therefore P(k+1) is true.

$b = 0$ Then

$$k = 4a + 1$$

We cannot use anymore $1 = 8-7$, as this would introduce a 7 with a negative coefficient. We note however that $1 = 21 - 20 = 3 \times 7 - 5 \times 4$. Therefore,

$$k = 4a + 3 \times 7 - 5 \times 4 = 4(a - 5) + 3 \times 7$$

Let $c = a - 5$ and $d = 3$. Obviously, $d \geq 0$. We note that since $k \geq 18$, and k is in the form $4a$, the smallest possible value for a is 5... therefore $c \geq 0$. We have therefore found two positive (or 0) integers (c, d) such that $k + 1 = 4c + 7d$. Therefore P(k+1) is true..

Therefore, in all cases, $P(k + 1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all n. Note that the proof by induction shows us that a solution exists, but does not show us how to get that solution! This is a case of a non-constructive proof.