Finals: Solutions

ECS 20 (Fall 2009)

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Part I: Logic

Exercise 1

On a distant island, every inhabitant is either a Knight or Knave. Knights only tell the truth. Knaves only tell lies - everything said by a Knave is false. You meet three inhabitants: A, B and C. A says, C is not a Knave. B says, C and A are both Knights. C says, A is a Knight or B is a Knave. Which, if any, are Knights? Which, if any, are Knaves?

The easiest is to use a table: K means knight, N means knave, T means that the person said the truth, and F that the person said a lie.

Line	A	B	C	A says	B says	C says
1	K	K	K	T	T	T
2	K	K	N	F	F	T
3	K	N	K	T	T	T
4	K	N	N	F	F	T
5	N	K	K	T	F	F
6	N	K	N	F	F	F
7	N	N	K	T	F	T
8	N	N	N	F	F	T

Now we note that:

- line 1 is consistent.
- line 2 cannot be correct: B would be a knight that lies and C a knave that tells the truth
- line 3 cannot be correct: B would be a knave that tells the truth.
- line 4 cannot be correct: A would be a knight that lies and C would be a knave that tells the truth.
- line 5 cannot be correct: A would be a knave that tells the truth and B and C would be knights that lie.

- line 6 cannot be correct: B would be a knight that lies.
- line 7 cannot be correct: A would be a knave that tells the truth.
- line 8 cannot be correct: C would be a knave that tells the truth.

Line 1 is the only valid possibility, i.e. A, B and C are all knights.

Exercise 2

Show that the proposition $P = [(r \vee p) \to (r \vee q)] \leftrightarrow [r \vee (p \to q)]$ is a tautology.

We use a truth table:

p	q	r	$r \vee p$	$r \vee q$	$(r \vee p) \to (r \vee q)$	$p \to q$	$r \vee (p \to q)$	P
T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T	T
T	F	T	T	T	T	F	T	T
T	F	F	T	F	F	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	F	F	T	T	T	T

Therefore P is a tautology.

Exercise 3

Let us play a logical game. You find yourself in front of three rooms whose doors are closed. One of these rooms contains a Lady, another a Tiger and the third room in empty. There is one sign on each door; you are told that the sign on the door of the room containing the Lady is true, the sign on the door of the room with the Tiger is false, and the sign on the door of the empty room could be either true or false. Here are the signs:

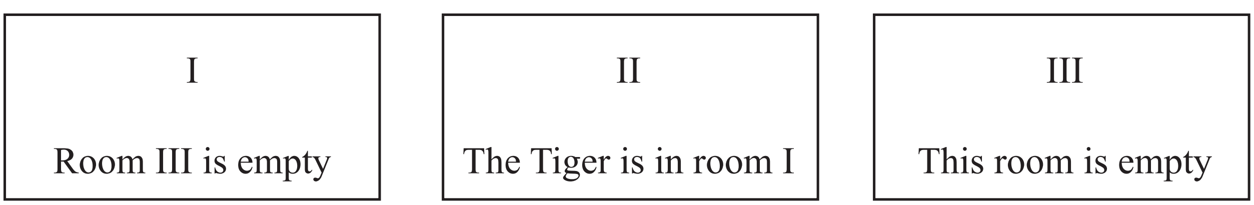

Figure 1: The three rooms and their signs

We solve this problem using a table, just like for the knights and knaves problem. Let us define the symbol LA, TI and EM for the room containing the Lady, the Tiger and being Empty, respectively. We do not know in which order these rooms are, but we do know we have all three (i.e. we do not have 2 rooms containing a Tiger for example): there are 6 different ways to organize these three rooms. For each way, we analyze the three signs given and indicate if they are true (T) or false (F):

Line	Room I	Room II	Room III	Sign I	Sign II	sign III
1	LA	TI	EM	T	F	T
2	LA	EM	TI	F	F	F
3	TI	LA	EM	T	T	T
4	TI	EM	LA	F	T	F
5	EM	LA	TI	F	F	F
6	EM	TI	LA	F	F	F

Now we note that:

- line 1 is consistent.
- line 2 cannot be correct: one of the signs must be true as one room contains the Lady
- line 3 cannot be correct: one of the signs must be false as one room contains the Tiger
- line 4 cannot be correct: the sign on the Lady's room would be false.
- line 5 cannot be correct: one of the signs must be true as one room contains the Lady
- line 6 cannot be correct: one of the signs must be true as one room contains the Lady

Line 1 is the only valid possibility, therefore the Lady is in room I, the tiger in room II, and room III is empty.

Part II : Proofs & Number Theory

Exercise 1

Prove or disprove that if p is prime, then $3p+1$ is prime.

For $p=3$, prime, $3p+1=10$, which is not prime. We found a counter-example: the proposition is not true.

Exercise 2

Show that the sum of any three consecutive perfect cubes is divisible by 9 (Note: a perfect cube is a number that can be written in the form n^3 where n is an integer. The three numbers $(n-1)^3$, n^3 and $(n+1)^3$ are three consecutive perfect cubes. (Hint: Start by showing that $n^3+2n \equiv 0[3]$ for all integer n)

We start with the proposition given in the hint:

$n^3+2n \equiv 0[3]$ for all integer n

There are many ways to prove that this is true: induction, direct proof,... I will use the shortest proof, but all other (valid) proofs are equally good.

Let n be an integer. Since 3 is prime, according to Fermat's little theorem:

$$n^3 \equiv n[3]$$

Therefore

$$n^3 + 2n \equiv 3n \equiv 0[3]$$

Therefore $n^3 + 2n$ is a multiple of 3 for all integers n.
Now let us compute the sum of three consecutive perfect cubes:

$$\begin{aligned}(n-1)^3 + n^3 + (n+1)^3 &= n^3 - 3n^2 + 3n - 1 + n^3 + n^3 + 3n^2 + 3n + 1 \\ &= 3n^3 + 6n \\ &= 3(n^3 + 2n)\end{aligned}$$

Based on the hint, we know that $n^3 + 2n$ is a multiple of 3: therefore there exists an integer k such that $n^3 + 2n = 3k$. Therefore,

$$(n-1)^3 + n^3 + (n+1)^3 = 9q$$

We conclude that the sum of three consecutive perfect cube is always divisible by 9.

Exercise 3

n is a positive integer. Prove that $12|(n^2 - 1)$ if $\gcd(n, 6) = 1$ (Hint: write $n = 6k + l$, and show that if $\gcd(n, 6) = 1$, then $l = 1$ or $l = 5$).

We use a direct proof.
Let n be a positive integer such that $\gcd(n, 6) = 1$. Let us consider the division of n by 6:

$$n = 6q + r$$

where r is the remainder that belongs to the set $\{0, 1, 2, 3, 4, 5\}$.
We note that:

- r cannot be equal to 0: n would be divisible by 6 and $\gcd(n, 6) = 6$.
- r cannot be equal to 2 or 4: n would be even, and $\gcd(n, 6) \geq 2$.
- r cannot be equal to 3: n would be a multiple of 3, and $\gcd(n, 6) = 3$.

Therefore, $n = 6q + 1$ or $n = 6q + 5$. Let us consider the two cases:

- $n = 6q + 1$. Then $n^2 - 1 = 36q^2 + 12q = 12(3q^2 + q)$ and $12|(n^2 - 1)$
- $n = 6q + 5$. Then $n^2 - 1 = 36q^2 + 12q + 24 = 12(3q^2 + q + 2)$ and $12|(n^2 - 1)$

Therefore $12|(n^2 - 1)$ for all n such that $\gcd(n, 6) = 1$.

Part III : Proof by Induction

Exercise 1

We want to Prove : $P(n)$ is true, for all $n \geq 1$, where
$P(n)$: $\sum_{i=1}^{n} \frac{2}{3^i} = 1 - \frac{1}{3^n}$
Let us define $\text{LHS}(n) = \sum_{i=1}^{n} \frac{2}{3^i}$, and $\text{RHS}(n) = 1 - \frac{1}{3^n}$.

- *Basis step:* We want to prove $P(1)$ is true.
  $\text{LHS}(1) = \frac{2}{3}$,
  and
  $\text{RHS}(1) = 1 - \frac{1}{3} = \frac{2}{3}$.
  Therefore, $\text{LHS}(1) = \text{RHS}(1)$: $P(1)$ is true.

- *Inductive step:* Let $P(k)$ be true for an integer $k \geq 1$, which means $\text{LHS}(k) = \text{RHS}(k)$. To prove that $P(k+1)$ is true, we prove that $\text{LHS}(k+1) = \text{RHS}(k+1)$. Let us compute $\text{LHS}(k+1)$:

$$\begin{aligned}
\text{LHS}(k+1) &= \sum_{i=1}^{k+1} \frac{2}{3} \\
&= \sum_{i=1}^{k} \frac{2}{3^i} + \frac{2}{3^{n+1}} \\
&= \text{LHS}(n) + \frac{2}{3^{n+1}} \\
&= \text{RHS}(n) + \frac{2}{3^{n+1}} \\
&= 1 - \frac{1}{3^n} + \frac{2}{3^{n+1}} \\
&= 1 - \frac{3}{3^{n+1}} + \frac{2}{3^{n+1}} \\
&= 1 - \frac{1}{3^{n+1}}
\end{aligned}$$

  and

$$\text{RHS}(k+1) = 1 - \frac{1}{3^{n+1}}$$

  Therefore $\text{LHS}(k+1) = \text{RHS}(k+1)$, i.e. $P(k+1)$ is true.

According to the principle of mathematical induction, we can conclude that $\sum_{i=1}^{n} \frac{2}{3^i} = 1 - \frac{1}{3^n}$ for all $n \geq 1$.

Exercise 2

Prove by induction that every number greater that 7 is the sum of a nonnegative integer multiple of 3 and a nonnegative integer multiple of 5.

Let us rewrite this as:

$P(n)$: For all $n \geq 8$, there exist two integers $k \geq 0$ and $l \geq 0$ such that $n = 3k + 5l$. We prove it by induction.

- *basis step*: For $n = 8$, we can set $k = 1$ and $l = 1$: $8 = 3 * 1 + 5 * 1$. $P(8)$ is true.
- *Inductive step*. Let $P(n)$ be true for an integer $n \geq 8$, i.e. there exist $k \geq 0$ and $l \geq 0$ such that $n = 3 * k + 5 * l$. We want to decompose $n + 1$.

  Notice that: $n+1 = 3*k+5*l+1 = 3*(k+2)-6+5*(l-1)+5+1 = 3*(k+2)+5*(l-1)$. $k + 2$ is nonnegative; however, $l - 1$ may be negative if $l = 0$. We study two cases:

  - $l = 0$ then $n = 3k$, i.e. n is a multiple of 3. Since n is greater than 7, $k \geq 3$.
    Notice that $n + 1 = 3k + 1 = 3(k - 3) + 2 * 5$. Since $k \geq 3$, $k - 3 \geq 0$: we have found two nonnegative integers $m = k - 3$ and $p = 2$ such that $n + 1 = 3m + 5p$.
  - $l > 0$ then $l - 1 \geq 0$. Therefore we have found two nonnegative integers $m = k + 2$ and $n = l - 1$ such that $n + 1 = 3m + 5p$.

  In all cases, $P(n + 1)$ is true.

According to the principle of mathematical induction, we can conclude that for all $n \geq 8$, there exist two integers $k \geq 0$ and $l \geq 0$ such that $n = 3k + 5l$.

Exercise 3

Prove by induction that $2^{n+1} > n^2 + 1$ for all $n \geq 2$.
Let us define $\text{LHS}(n) = 2^{n+1}$ and $\text{RHS}(n) = n^2 + 1$.

- *Basis case:* Let us prove that $P(2)$ is true:
  $\text{LHS}(2) = 2^3 = 8$
  $\text{RHS}(2) = 2^2 + 1 = 5$.
  Therefore $\text{LHS}(2) > \text{RHS}(2)$. P(2) is true.
- *Inductive step:* Let us assume that $P(n)$ is true. This means that $\text{LHS}(n) > \text{RHS}(n)$. We want to prove $P(n + 1) : \text{LHS}(n + 1) > \text{RHS}(n + 1)$ with $\text{LHS}(n + 1) = 2^{n+2}$ and $\text{RHS}(n + 1) = (n + 1)^2 + 1$.

$$\begin{aligned} \text{LHS}(n+1) &= 2^{n+2} \\ &= 2 * 2^{n+1} \\ &= 2 * \text{LHS}(n) \\ &> 2 * \text{RHS}(n) \\ &> 2(n^2+1) \end{aligned}$$

  Let us rewrite:

$$\begin{aligned} 2(n^2+1) &= n^2 + n^2 + 2 \\ &= n^2 + 2n + 2 + n^2 - 2n \\ &= (n+1)^2 + 1 + n^2 - 2n \\ &= (n+1)^2 + 1 + n(n-2) \end{aligned}$$

Since $n \geq 2$, $n(n-2) \geq 0$, therefore $2(n^2+1) \geq (n+1)^2+1$.
Replacing above, we get:

$$\begin{aligned} \text{LHS}(n+1) &> (n+1)^2+1 \\ &> \text{RHS}(n+1) \end{aligned}$$

Therefore $P(n+1)$ is true.

According to the principle of mathematical induction, we can conclude that $P(n)$ is true for all $n \geq 2$.

Exercise 4

4) Let fn be the n-th Fibonacci number. we want to show that f_{3n} is even for all $n \geq 1$.
We use a proof by induction.

- *Basis step:* $f_3 = f_2 + f_1 = 1 + 1 = 2$ which is even. The proposition is true for $n = 1$.
- *Inductive step:*. We suppose that f_{3n} is even for $n \geq 1$. Then:
  $f_{3n+3} = f_{3n+2} + f_{3n+1} = f_{3n+1} + f_{3n} + f_{3n+1} = 2 * f_{3n+1} + f_{3n}$.
  $2 * f_{3n+1}$ is even and f_{3n} is even by hypothesis; therefore $f_{3(n+1)}$ is even.

According to the principle of mathematical induction, we can conclude that f_{3n} is even for all $n \geq 1$.

Part IV : Counting. Pigeonhole Principle

Exercise 1

Prove that if 6 distinct numbers are selected from $\{1, 2, \ldots, 9, 10\}$, then there will be at least two that are consecutive.

This is a pigeonhole principle problem. We must define the boxes and the objects:

- *objects*: the 6 distinct numbers
- *boxes*: We consider the five boxes formed by consecutive numbers in the set:
  (1,2) (3,4) (5,6) (7,8) (9,10)

Since the boxes are disjoint, to each object we can assign a unique box. There are 6 objects and five boxes: according the the pigeonhole principle, there is (at least) one box that contains two objects. These two objects are the two consecutive integers we are looking for.

Exercise 2

If 5 points are selected at random from the interior of the unit circle, then there are 2 points whose distance is less than $\sqrt{2}$.
Let us divide the interior of the unit circle into 4 equal quadrants.

- *objects*: the 5 points
- *boxes*: the four quadrants

According the the pigeonhole principle, there is (at least) one box (quadrant) that contains two objects (points). The distance between these two points is less than $\sqrt{2}$.

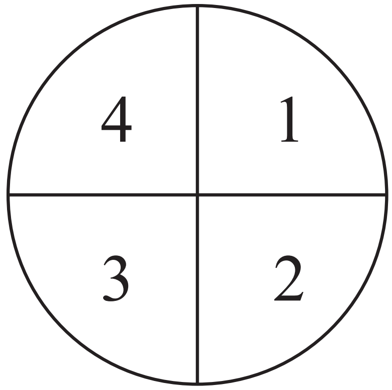

Figure 2: The unit circle divided into 4 quadrants

Exercise 3

Given any five integers, there will be three for which the sum of the squares of those integers is divisible by 3.
We use the pigeonhole principle:

- *objects*: the 5 integers
- *boxes*: two boxes: one that says "divisible by 3", the other that says "not divisible by 3"

The two boxes are disjoint and there is no ambiguity in assigning a box to one of the integers. According to the pigeonhole principle, one of these two boxes contain at least 3 elements. Let us refer to these three integers as a, b and c. We check the two cases:

- *Case 1*: The three integers belong to the box "divisible by 3". Then a, b and c are divisible by 3; therefore a^2, b^2 and c^2 are divisible by 3 and a fortiori $a^2 + b^2 + c2$ is divisible by 3.
- *case 2*: The three integers belong to the box "not divisible by 3".

  Since a is not divisible by 3 and 3 is prime, according to Fermat's little theorem, $a^2 \equiv 1[3]$. Similarly, $b^2 \equiv 1[3]$ and $c^2 \equiv 1[3]$. Therefore $a^2 + b^2 + c^2 \equiv 3 \equiv 0[3]$.

In all cases, $a^2 + b^2 + c^2$ is divisible by 3.

Exercise 4

A combination lock has three numbers in the combination, each in the range 1 to 40

a) How many different combinations are there?
   There are 40 possible values for each of the three numbers; using the product rules, the total number of combinations is:
   T = 40x40x40 = 64,000

b) How many of the combinations have no duplicate numbers? There are 40 possible values for the first number, 39 for the second number, and 38 for the last number as we do not want duplicates. Therefore, the total number of combinations with no duplicates is:
   D= 40x39x38 = 59280

c) How many of the combinations have exactly two of the three numbers matching?
The combination is defined by two numbers: the one that is duplicated, and a second number that is not equal to the first one. In addition, there are three possibilities for positioning the number that is not duplicated. Therefore, the total number of combinations that has exactly two of the three numbers matching is:
D2 = 3x40x39 = 4680.
An alternate solution:
There are three types of combinations: those without any duplicate (D), those with exactly two of the three numbers matching (D2), and those that have all three numbers matching (D3). It is easy to see that D3 = 40. Therefore:
T = D + D2 + D3
D2 = T - D - D3 = 64000 - 59280 - 40 = 4680.