Midterm 1: Solutions

ECS20 (Winter 2019)

Part I: logic

1) Let p, q, and r be three propositions. Using truth tables or logical equivalences, indicate which (if any) of the propositions below are tautologies, contradictions, or neither.

a) $A = (\neg p \vee q) \wedge [q \rightarrow (\neg r \wedge \neg p)] \wedge (p \vee r)$

We solve this problem by using a truth table

p	q	r	$\neg p \vee q$	$\neg r \wedge \neg p$	$q \rightarrow (\neg r \wedge \neg p)$	$p \vee r$	A
T	T	T	T	F	F	T	F
T	T	F	T	F	F	T	F
T	F	T	F	F	T	T	F
T	F	F	F	F	T	T	F
F	T	T	T	F	F	T	F
F	T	F	T	T	T	F	F
F	F	T	T	F	T	T	T
F	F	F	T	T	T	F	F

The proposition A is neither tautology, nor a contradiction.

b) $[(p \rightarrow q) \wedge (q \rightarrow \neg p)] \rightarrow p \leftrightarrow p$

Let $A = [(p \rightarrow q) \wedge (q \rightarrow \neg p)]$, $B = A \rightarrow p$ and $C = B \leftrightarrow p$. Let us build the truth table:

p	q	$\neg p$	$q \rightarrow q$	$q \rightarrow \neg p$	A	B	C
T	T	F	T	F	F	T	T
T	F	F	F	T	F	T	T
F	T	T	T	T	T	F	T
F	F	T	T	T	T	F	T

The proposition C is a tautology.

2) The Lady and the Tiger

Let us play a logical game. You find yourself in front of two rooms whose doors are closed. Behind each door, there could be a Lady or a Tiger. There is one sign on each door; you are told that if the first room contains a Lady, then the sign is true, but if a Tiger is in it, the sign is false. In room 2, the situation is the opposite: a Lady in room 2 means the sign is false, while a Tiger in room 2 means the sign is true. The signs on both doors say "Both rooms contain Ladies". Can you say what is behind each door? Justify your answer.

Let us build the table for the possible options for the two doors. We then check the validity of the two statements on the door, and finally check the consistency of the truth values for those statements with what we know about their validity.

Line	Room I	Room II	Sign on Room I	Sign on Room II	Compatibility
1	Lady	Lady	T	T	No
2	Lady	Tiger	F	F	No
3	Tiger	Lady	F	F	Yes
4	Tiger	Tiger	F	F	No

Justification:

Line 1 : There is a Lady in room II; the sign on that room should then be false, but it is true: incompatibility

Line 2 : There is a Lady in room I; the sign on that room should then be true, but it is false: incompatibility

Line 4 : There is a Tiger in room II; the sign on that room should then be true, but it is false: incompatibility

Therefore only line 3 is compatible: there is a Tiger in room 1 and a Lady in room 2!

Part II: proofs

Exercise 1

Let m and n be two integers. Show that if $m > 0$ and $n \leq -2$, then $m^2 + mn + n^2 \geq 0$.

We use a direct proof. Let p be the proposition $m > 0$ and $n \leq -2$, and let q be the proposition $m^2 + mn + n^2 \geq 0$. To show $p \to q$, we will show that if p is true, then q is also true.

Let m and n be two integers.

Hypothesis: p is true. Therefore $m > 0$ and $n \leq -2$. As m is (strictly) positive, we can multiply $n \leq -2$ by m without changing the sense of the inequality; then $mn \leq -2m$. As m is strictly positive, $-2m$ is strictly negative. Therefore:

$$\begin{aligned} mn &\leq -2m \\ -2m &< 0 \end{aligned}$$

i.e. $mn < 0$.

Let us consider now:

$$\begin{aligned} m^2 + mn + n^2 &= m^2 + 2mn + n^2 - mn \\ &= (m+n)^2 - mn \end{aligned}$$

Note that $(m+n)^2$ is positive and $-mn$ is also positive, as mn is negative. $m^2 + mn + n^2$ is the sum of two positive numbers; it is positive. Therefore q is true, and the property is true.

Exercise 2

a) Let x be a real number, with $x \neq 0$. Use a proof by contradiction to show that if $x + \frac{1}{x} < 2$, then $x < 0$.

Let p be the proposition: $x + \frac{1}{x} < 2$ and let q be the proposition $x < 0$. We want to show $p \rightarrow q$ using a proof by contradiction. We suppose that $p \rightarrow q$ is false, which is equivalent to say that $\neg p \vee q$ is false, i.e. that p is true AND q is false.

p is true: $x + \frac{1}{x} < 2$; q is false, therefore $x \geq 0$.

Since $x \geq 0$, we can multiply $x + \frac{1}{x} < 2$ by x without changing the sense of the inequality. We obtain:

$$x^2 + 1 < 2x \tag{1}$$

which we can rewrite as:

$$\begin{aligned} x^2 + 1 - 2x &< 0 \\ (x-1)^2 &< 0 \end{aligned}$$

However, the term on the left is a square and should be positive: we have reached a contradiction. Therefore, the hypothesis that $p \rightarrow q$ is false, is false, and $p \rightarrow q$ is true!

b) Let x be a real number, with $x \neq 0$. Use a direct proof to show that if $x + \frac{1}{x} < 2$, then $x < 0$.

Let p be the proposition: $x + \frac{1}{x} < 2$ and let q be the proposition $x < 0$. We want to show $p \rightarrow q$ using a direct proof. We suppose that p is true and we want to show that q is true.

p is true: $x + \frac{1}{x} < 2$; we put all terms on the left, and set all terms with the same denominator:

$$\begin{aligned} x + \frac{1}{x} - 2 &< 0 \\ \frac{x^2 + 1 - 2x}{x} &< 0 \\ \frac{(x-1)^2}{x} &< 0 \end{aligned}$$

The term on the left is the ratio of a square with x: as the numerator is positive, the denominator has to be negative: $x < 0$. Therefore q is true, and the proposition $p \rightarrow q$ is true.

Exercise 3

Let p be a natural number. Show that $2^{\frac{1}{4}}$ is irrational.

We use a proof by contradiction: let us suppose that $2^{\frac{1}{4}}$ is a rational number. There exists two integers a and b, with $b \neq 0$ such that

$$2^{\frac{1}{4}} = \frac{a}{b} \tag{2}$$

After raising this equation to the power 2, we get:

$$\sqrt{2} = \frac{a^2}{b^2} \tag{3}$$

As a and b are integers; a^2 and b^2 are integers, with $b^2 \neq 0$. The equation above would then mean that $\sqrt{2}$ is rational; this is not true. Therefore $2^{\frac{1}{4}}$ is irrational.

Extra credit

Let a and b be two integers. Use a direct proof to show that if $a^2 + b^2$ is even, then $a + b$ is even

Let a and b be two integers. Let p be the proposition $a^2 + b^2$ is even and let q be the proposition $a + b$ is even.

We are asked to use a direct proof...

Hypothesis: p is true. Then $a^2 + b^2$ is even, i.e. there exists an integer k such that $a^2 + b^2 = 2k$. Note that

$$\begin{aligned}(a+b)^2 &= a^2 + b^2 + 2ab \\ &= 2k + 2ab \\ &= 2(k + ab)\end{aligned}$$

As $k + ab$ is an integer, $(a + b)^2$ is even, and therefore $a + b$ is even.