Midterm 2: Solutions

ECS20 (Winter 2019)

February, 2019

Part I: Proofs

Let a and b be two real numbers with $a \neq 0$ and $b \neq 0$. Use a proof by contradiction to show that if $ab > 0$, then $\frac{a}{b} + \frac{b}{a} \geq 2$.

Let:
p: $ab > 0$
q: $\frac{a}{b} + \frac{b}{a} \geq 2$

and let A be the proposition $p \rightarrow q$. We want to show that $\forall n \in \mathbb{N}$, A is true. We use a proof by contradiction, i.e. we suppose that what we want to show is false, namely that $\exists n \in \mathbb{N}$, A is not true, i.e. $\exists n \in \mathbb{N}$, p is true AND q is false.

p is true: $ab > 0$. Similarly, as q is false, $\frac{a}{b} + \frac{b}{a} < 2$. As $ab > 0$, we can multiply this inequality by ab without changing its sense; we get:

$$a^2 + b^2 < 2ab$$

which gives

$$a^2 + b^2 - 2ab < 0$$

i.e.

$$(a-b)^2 < 0$$

However, $(a-b)^2$ is a square, and therefore $(a-b)^2 \geq 0$. we have reached a contradiction. The proposition A is therefore true.

Part II: Sets

Let A, B, and C be three sets in a domain D. Consider the following possible equalities, $(A \bigcap B) - C = (A - C) \bigcap (B - C)$ and $C - (A \bigcap B) = (C - A) \bigcap (C - B)$. Show that one of these equalities is always true, but the other can be false (for the latter, give an example).

We check the first proposition. We can use a proof by membership table. I will use the set identities. Let $LHS = A \bigcap B - C$ and $RHS = (A - C) \bigcap (B - C)$.

Then:

$$\begin{aligned} LHS &= (A \bigcap B) \bigcap \overline{C} \\ &= A \bigcap B \bigcap \overline{C} \end{aligned}$$

and

$$\begin{aligned} RHS &= (A-C)\bigcap(B-C) \\ &= A\bigcap\overline{C}\bigcap B\bigcap\overline{C} \\ &= A\bigcap B\bigcap\overline{C} \\ &= LHS \end{aligned}$$

Therefore the two sets LHS and RHS are equal.

The second proposition can therefore be false. Let us build the membership table for $LHS = C - A\bigcap B$ and $RHS = (C-A)\bigcap(C-B)$.

A	B	C	$A\bigcap B$	LHS	$C-A$	$C-B$	RHS
1	1	1	1	0	0	0	0
1	1	0	1	0	0	0	0
1	0	1	0	1	0	1	0
1	0	0	0	0	0	0	0
0	1	1	0	1	1	0	0
0	1	0	0	0	0	0	0
0	0	1	0	1	1	1	1
0	0	0	0	0	0	0	0

Notice that the membership values for LHS and RHS do not match: see the two rows with the red values. This happens when there is a value in C and A, but not in B, or a value in C and B, but not in A. This allows us to construct a counter example. Let for example $A = \{1,2,3\}$, $B = \{3,4,5\}$, and $C = \{1,5,7\}$. Then:

$$\begin{aligned} LHS &= C-(A\bigcap B) \\ &= \{1,5,7\}-\{3\} \\ &= \{1,5,7\} \end{aligned}$$

and

$$\begin{aligned} LHS &= (C-A)\bigcap(C-B) \\ &= \{5,7\}\bigcap\{1,7\} \\ &= \{7\} \end{aligned}$$

In this case, $LHS \neq RHS$.

2. Let A, B, and C be three sets in a domain D; we assume that $A - C \subset B$. x is an element of D. Show that if $x \in A - B$, then $x \in C$.

We need to prove an implication of the form $p \rightarrow q$, where:

$p : x \in A - B$
$q : x \in C$.

We will use a proof by contradiction.

Hypothesis: $p \to q$ is false, i.e. p is true and $\neg q$ is true, namely $x \in A - B$, and $x \notin C$. We also know that $A - C \subset B$.
Let x be an element of D with $x \in A - B$. By definition of the difference between two sets, $x \in A$, and $x \notin B$. Since $x \in A$ and $x \notin C$, $x \in A - C$. As $A - C \subset B$, we have $x \in B$. This leads to $x \notin B$ and $x \in B$, which is a contradiction.
Therefore, the hypothesis that $p \to q$ is false, is false, and $p \to q$ is true. This concludes the proof.

Part III: functions

1) *Let n and m be two integers. Solve $\lfloor \frac{n+m}{2} \rfloor + \lfloor \frac{n-m+1}{2} \rfloor = n$.*

   Let n and m be two integers. Let us define $LHS = \lfloor \frac{n+m}{2} \rfloor + \lfloor \frac{n-m+1}{2} \rfloor$. Notice that as we consider division by 2, we will consider parity, and use a proof by case. We consider the parity of $n + m$:

   a) $n + m$ is even. There exists an integer k such that $n + m = 2k$. We note also the $n - m = n + m - 2m = 2k - 2m$. Then

$$\begin{aligned}
LHS &= \lfloor \frac{n+m}{2} \rfloor + \lfloor \frac{n-m+1}{2} \rfloor \\
&= \lfloor \frac{2k}{2} \rfloor + \lfloor \frac{2k-2m+1}{2} \rfloor \\
&= \lfloor k \rfloor + \lfloor k - m + \frac{1}{2} \rfloor \\
&= k + k - m + \lfloor \frac{1}{2} \rfloor \\
&= 2k - m \\
&= n + m - m \\
&= n
\end{aligned}$$

   b) $n + m$ is odd. There exists an integer k such that $n + m = 2k + 1$. We note also the $n - m = n + m - 2m = 2k + 1 - 2m$. Then

$$\begin{aligned}
LHS &= \lfloor \frac{n+m}{2} \rfloor + \lfloor \frac{n-m+1}{2} \rfloor \\
&= \lfloor \frac{2k+1}{2} \rfloor + \lfloor \frac{2k-2m+2}{2} \rfloor \\
&= \lfloor k + \frac{1}{2} \rfloor + \lfloor k - m + 1 \rfloor \\
&= k + k - m + 1 \rfloor \\
&= 2k + 1 - m \\
&= n + m - m \\
&= n
\end{aligned}$$

   In all cases, we have $LHS = n$.

2) *Let x be a real number, and let n be a natural number. Show that* $\left\lfloor \frac{x+3}{n} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor + 3}{n} \right\rfloor$.

Let us define $l = \left\lfloor \frac{x+3}{n} \right\rfloor$ and $m = \lfloor x \rfloor$ (l and m are both integers). By definition of floor, we have the two properties:

$l \leq \frac{x+3}{n} < l + 1$

and

$m \leq x < m + 1$

Let us multiply the first inequality by n:

$nl \leq x + 3 < n(l + 1)$

Now we subtract 3 from the same inequalities:

$nl - 3 \leq x < n(l + 1) - 3$

We notice that:

$m \leq x$ and $x < n(l + 1) - 3$; therefore $m < n(l + 1) - 3$.

$m \leq x$ and $nl - 3 \leq x$. Therefore m and $nl - 3$ are two integers smaller than x. By definition of floor, m is the largest integer smaller that x. Therefore $nl - 3 \leq m$.

Combining those two inequalities, we get $nl - 3 \leq m < n(l + 1) - 3$. After addition of 3, and division by n, $l \leq \frac{m+3}{n} < l + 1$. Therefore l is the floor of $\frac{m+3}{n}$. Replacing l and m by their values, we get:

$$l = \left\lfloor \frac{x+3}{n} \right\rfloor = \left\lfloor \frac{m+3}{n} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor + 3}{n} \right\rfloor$$

The property is therefore true.

Part III: Proofs

Show that $\{p| \quad p \quad \text{is prime}\} \bigcap \{k^2 - 1 | k \in \mathbf{N}\} = 3$.

Let $A = \{p| \quad p \quad \text{is prime}\}$ and $B = \{k^2 - 1 | k \in \mathbf{Z}\}$. Both sets are sets of integers. An element n of $A \bigcap B$ satisfy the two properties:

a) n is prime

b) There exists $k \in \mathbb{Z}$ such that $n = k^2 - 1$

Notice that $n = k^2 - 1 = (k - 1)(k + 1)$. As n is prime, and $k \geq 1$, we must have $k - 1 = 1$, i.e. $k = 2$. Therefore $n = 3$, and $A \bigcap B = \{3\}$.

Extra credit

Let x be a real number. Solve $\frac{x-1}{2} = \lfloor \frac{x}{2} \rfloor - \lfloor \frac{x+1}{2} \rfloor$.

We notice first that $\frac{x-1}{2}$ must be an integer, as it is the difference between two floors. Notice also that $\frac{x+1}{2} = \frac{x-1}{2} + 1$ and therefore $\frac{x+1}{2}$ is also an integer. Replacing in the equation above, we get:

$$
\begin{aligned}
\frac{x-1}{2} + \lfloor \frac{x+1}{2} \rfloor &= \lfloor \frac{x}{2} \rfloor \\
\frac{x-1}{2} + \frac{x+1}{2} &= \lfloor \frac{x}{2} \rfloor \\
x &= \lfloor \frac{x}{2} \rfloor
\end{aligned}
$$

Therefore x is also an integer. We consider 2 cases:

a) x is even. There exists an integer k such that $x = 2k$. The equation becomes:

$$
\begin{aligned}
2k &= \lfloor \frac{x}{2} \rfloor \\
&= \lfloor \frac{2k}{2} \rfloor \\
&= k
\end{aligned}
$$

This would mean that $k = 0$, i.e. $x = 0$. But then $\frac{x-1}{2}$ would not be an integer. There are no even solutions to the equation.

b) x is odd. There exists an integer k such that $x = 2k + 1$. The equation becomes:

$$
\begin{aligned}
2k+1 &= \lfloor \frac{x}{2} \rfloor \\
&= \lfloor \frac{2k+1}{2} \rfloor \\
&= k
\end{aligned}
$$

This equation has $k = -1$ for solution, in which case $x = -1$.

Verification:

a) $\frac{x-1}{2} = -1$

b) $\lfloor \frac{x}{2} \rfloor - \lfloor \frac{x+1}{2} \rfloor = \lfloor \frac{-1}{2} \rfloor - \lfloor \frac{-1+1}{2} \rfloor = -1$

Therefore the only solution to the equation is $x = -1$.