One useful fact is that the leading term of a polynomial determines its order. For example, if $f(x) = 3x^5 + x^4 + 17x^3 + 2$, then $f(x)$ is of order x^5. This is stated in Theorem 4, whose proof is left as Exercise 50.

THEOREM 4 Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_n$ are real numbers with $a_n \neq 0$. Then $f(x)$ is of order x^n.

EXAMPLE 13 The polynomials $3x^8 + 10x^7 + 221x^2 + 1444$, $x^{19} - 18x^4 - 10{,}112$, and $-x^{99} + 40{,}001x^{98} + 100{,}003x$ are of orders x^8, x^{19}, and x^{99}, respectively. ◀

Unfortunately, as Knuth observed, big-O notation is often used by careless writers and speakers as if it had the same meaning as big-Theta notation. Keep this in mind when you see big-O notation used. The recent trend has been to use big-Theta notation whenever both upper and lower bounds on the size of a function are needed.

Exercises

In Exercises 1–14, to establish a big-O relationship, find witnesses C and k such that $|f(x)| \leq C|g(x)|$ whenever $x > k$.

1. Determine whether each of these functions is $O(x)$.
  a) $f(x) = 10$ **b)** $f(x) = 3x + 7$
  c) $f(x) = x^2 + x + 1$ **d)** $f(x) = 5 \log x$
  e) $f(x) = \lfloor x \rfloor$ **f)** $f(x) = \lceil x/2 \rceil$

2. Determine whether each of these functions is $O(x^2)$.
  a) $f(x) = 17x + 11$ **b)** $f(x) = x^2 + 1000$
  c) $f(x) = x \log x$ **d)** $f(x) = x^4/2$
  e) $f(x) = 2^x$ **f)** $f(x) = \lfloor x \rfloor \cdot \lceil x \rceil$

3. Use the definition of "$f(x)$ is $O(g(x))$" to show that $x^4 + 9x^3 + 4x + 7$ is $O(x^4)$.

4. Use the definition of "$f(x)$ is $O(g(x))$" to show that $2^x + 17$ is $O(3^x)$.

5. Show that $(x^2 + 1)/(x + 1)$ is $O(x)$.

6. Show that $(x^3 + 2x)/(2x + 1)$ is $O(x^2)$.

7. Find the least integer n such that $f(x)$ is $O(x^n)$ for each of these functions.
  a) $f(x) = 2x^3 + x^2 \log x$
  b) $f(x) = 3x^3 + (\log x)^4$
  c) $f(x) = (x^4 + x^2 + 1)/(x^3 + 1)$
  d) $f(x) = (x^4 + 5 \log x)/(x^4 + 1)$

8. Find the least integer n such that $f(x)$ is $O(x^n)$ for each of these functions.
  a) $f(x) = 2x^2 + x^3 \log x$
  b) $f(x) = 3x^5 + (\log x)^4$
  c) $f(x) = (x^4 + x^2 + 1)/(x^4 + 1)$
  d) $f(x) = (x^3 + 5 \log x)/(x^4 + 1)$

9. Show that $x^2 + 4x + 17$ is $O(x^3)$ but that x^3 is not $O(x^2 + 4x + 17)$.

10. Show that x^3 is $O(x^4)$ but that x^4 is not $O(x^3)$.

11. Show that $3x^4 + 1$ is $O(x^4/2)$ and $x^4/2$ is $O(3x^4 + 1)$.

12. Show that $x \log x$ is $O(x^2)$ but that x^2 is not $O(x \log x)$.

13. Show that 2^n is $O(3^n)$ but that 3^n is not $O(2^n)$. (Note that this is a special case of Exercise 60.)

14. Determine whether x^3 is $O(g(x))$ for each of these functions $g(x)$.
  a) $g(x) = x^2$ **b)** $g(x) = x^3$
  c) $g(x) = x^2 + x^3$ **d)** $g(x) = x^2 + x^4$
  e) $g(x) = 3^x$ **f)** $g(x) = x^3/2$

15. Explain what it means for a function to be $O(1)$.

16. Show that if $f(x)$ is $O(x)$, then $f(x)$ is $O(x^2)$.

17. Suppose that $f(x)$, $g(x)$, and $h(x)$ are functions such that $f(x)$ is $O(g(x))$ and $g(x)$ is $O(h(x))$. Show that $f(x)$ is $O(h(x))$.

18. Let k be a positive integer. Show that $1^k + 2^k + \cdots + n^k$ is $O(n^{k+1})$.

19. Determine whether each of the functions 2^{n+1} and 2^{2n} is $O(2^n)$.

20. Determine whether each of the functions $\log(n + 1)$ and $\log(n^2 + 1)$ is $O(\log n)$.

21. Arrange the functions $\sqrt{n}$, $1000 \log n$, $n \log n$, $2n!$, 2^n, 3^n, and $n^2/1{,}000{,}000$ in a list so that each function is big-O of the next function.

22. Arrange the function $(1.5)^n$, n^{100}, $(\log n)^3$, $\sqrt{n} \log n$, 10^n, $(n!)^2$, and $n^{99} + n^{98}$ in a list so that each function is big-O of the next function.

23. Suppose that you have two different algorithms for solving a problem. To solve a problem of size n, the first algorithm uses exactly $n(\log n)$ operations and the second algorithm uses exactly $n^{3/2}$ operations. As n grows, which algorithm uses fewer operations?

24. Suppose that you have two different algorithms for solving a problem. To solve a problem of size n, the first algorithm uses exactly $n^2 2^n$ operations and the second algorithm uses exactly $n!$ operations. As n grows, which algorithm uses fewer operations?

25. Give as good a big-O estimate as possible for each of these functions.

a) $(n^2 + 8)(n + 1)$ **b)** $(n \log n + n^2)(n^3 + 2)$

c) $(n! + 2^n)(n^3 + \log(n^2 + 1))$

26. Give a big-O estimate for each of these functions. For the function g in your estimate $f(x)$ is $O(g(x))$, use a simple function g of smallest order.

a) $(n^3+n^2 \log n)(\log n+1) + (17 \log n+19)(n^3+2)$

b) $(2^n + n^2)(n^3 + 3^n)$

c) $(n^n + n2^n + 5^n)(n! + 5^n)$

27. Give a big-O estimate for each of these functions. For the function g in your estimate that $f(x)$ is $O(g(x))$, use a simple function g of the smallest order.

a) $n \log(n^2 + 1) + n^2 \log n$

b) $(n \log n + 1)^2 + (\log n + 1)(n^2 + 1)$

c) $n^{2^n} + n^{n^2}$

28. For each function in Exercise 1, determine whether that function is $\Omega(x)$ and whether it is $\Theta(x)$.

29. For each function in Exercise 2, determine whether that function is $\Omega(x^2)$ and whether it is $\Theta(x^2)$.

30. Show that each of these pairs of functions are of the same order.

a) $3x + 7$, x

b) $2x^2 + x - 7$, x^2

c) $\lfloor x + 1/2 \rfloor$, x

d) $\log(x^2 + 1)$, $\log_2 x$

e) $\log_{10} x$, $\log_2 x$

31. Show that $f(x)$ is $\Theta(g(x))$ if and only if $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$.

32. Show that if $f(x)$ and $g(x)$ are functions from the set of real numbers to the set of real numbers, then $f(x)$ is $O(g(x))$ if and only if $g(x)$ is $\Omega(f(x))$.

33. Show that if $f(x)$ and $g(x)$ are functions from the set of real numbers to the set of real numbers, then $f(x)$ is $\Theta(g(x))$ if and only if there are positive constants k, C_1, and C_2 such that $C_1|g(x)| \le |f(x)| \le C_2|g(x)|$ whenever $x > k$.

34. **a)** Show that $3x^2 + x + 1$ is $\Theta(3x^2)$ by directly finding the constants k, C_1, and C_2 in Exercise 33.

b) Express the relationship in part (a) using a picture showing the functions $3x^2 + x + 1$, $C_1 \cdot 3x^2$, and $C_2 \cdot 3x^2$, and the constant k on the x-axis, where C_1, C_2, and k are the constants you found in part (a) to show that $3x^2 + x + 1$ is $\Theta(3x^2)$.

35. Express the relationship $f(x)$ is $\Theta(g(x))$ using a picture. Show the graphs of the functions $f(x)$, $C_1|g(x)|$, and $C_2|g(x)|$, as well as the constant k on the x-axis.

36. Explain what it means for a function to be $\Omega(1)$.

37. Explain what it means for a function to be $\Theta(1)$.

38. Give a big-O estimate of the product of the first n odd positive integers.

39. Show that if f and g are real-valued functions such that $f(x)$ is $O(g(x))$, then for every positive integer n, $f^n(x)$ is $O(g^n(x))$. [Note that $f^n(x) = f(x)^n$.]

40. Show that for all real numbers a and b with $a > 1$ and $b > 1$, if $f(x)$ is $O(\log_b x)$, then $f(x)$ is $O(\log_a x)$.

41. Suppose that $f(x)$ is $O(g(x))$ where f and g are increasing and unbounded functions. Show that $\log |f(x)|$ is $O(\log |g(x)|)$.

42. Suppose that $f(x)$ is $O(g(x))$. Does it follow that $2^{f(x)}$ is $O(2^{g(x)})$?

43. Let $f_1(x)$ and $f_2(x)$ be functions from the set of real numbers to the set of positive real numbers. Show that if $f_1(x)$ and $f_2(x)$ are both $\Theta(g(x))$, where $g(x)$ is a function from the set of real numbers to the set of positive real numbers, then $f_1(x) + f_2(x)$ is $\Theta(g(x))$. Is this still true if $f_1(x)$ and $f_2(x)$ can take negative values?

44. Suppose that $f(x)$, $g(x)$, and $h(x)$ are functions such that $f(x)$ is $\Theta(g(x))$ and $g(x)$ is $\Theta(h(x))$. Show that $f(x)$ is $\Theta(h(x))$.

45. If $f_1(x)$ and $f_2(x)$ are functions from the set of positive integers to the set of positive real numbers and $f_1(x)$ and $f_2(x)$ are both $\Theta(g(x))$, is $(f_1 - f_2)(x)$ also $\Theta(g(x))$? Either prove that it is or give a counterexample.

46. Show that if $f_1(x)$ and $f_2(x)$ are functions from the set of positive integers to the set of real numbers and $f_1(x)$ is $\Theta(g_1(x))$ and $f_2(x)$ is $\Theta(g_2(x))$, then $(f_1 f_2)(x)$ is $\Theta((g_1 g_2)(x))$.

47. Find functions f and g from the set of positive integers to the set of real numbers such that $f(n)$ is not $O(g(n))$ and $g(n)$ is not $O(f(n))$.

48. Express the relationship $f(x)$ is $\Omega(g(x))$ using a picture. Show the graphs of the functions $f(x)$ and $Cg(x)$, as well as the constant k on the real axis.

49. Show that if $f_1(x)$ is $\Theta(g_1(x))$, $f_2(x)$ is $\Theta(g_2(x))$, and $f_2(x) \neq 0$ and $g_2(x) \neq 0$ for all real numbers $x > 0$, then $(f_1/f_2)(x)$ is $\Theta((g_1/g_2)(x))$.

50. Show that if $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_{n-1}$, and a_n are real numbers and $a_n \neq 0$, then $f(x)$ is $\Theta(x^n)$.

Big-O, big-Theta, and big-Omega notation can be extended to functions in more than one variable. For example, the statement $f(x, y)$ is $O(g(x, y))$ means that there exist constants C, k_1, and k_2 such that $|f(x, y)| \le C|g(x, y)|$ whenever $x > k_1$ and $y > k_2$.

51. Define the statement $f(x, y)$ is $\Theta(g(x, y))$.

52. Define the statement $f(x, y)$ is $\Omega(g(x, y))$.

53. Show that $(x^2 + xy + x \log y)^3$ is $O(x^6 y^3)$.

54. Show that $x^5 y^3 + x^4 y^4 + x^3 y^5$ is $\Omega(x^3 y^3)$.

55. Show that $\lfloor xy \rfloor$ is $O(xy)$.

56. Show that $\lceil xy \rceil$ is $\Omega(xy)$.

57. (*Requires calculus*) Show that if $c > d > 0$, then n^d is $O(n^c)$, but n^c is not $O(n^d)$.

58. (*Requires calculus*) Show that if $b > 1$ and c and d are positive, then $(\log_b n)^c$ is $O(n^d)$, but n^d is not $O((\log_b n)^c)$.

59. (*Requires calculus*) Show that if d is positive and $b > 1$, then n^d is $O(b^n)$ but b^n is not $O(n^d)$.

60. (*Requires calculus*) Show that if $c > b > 1$, then b^n is $O(c^n)$ but c^n is not $O(b^n)$.

The following problems deal with another type of asymptotic notation, called **little-*o*** notation. Because little-o notation is based on the concept of limits, a knowledge of calculus is needed for these problems. We say that $f(x)$ is $o(g(x))$ [read $f(x)$ is "little-oh" of $g(x)$], when

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

61. (*Requires calculus*) Show that

a) x^2 is $o(x^3)$. **b)** $x \log x$ is $o(x^2)$.

c) x^2 is $o(2^x)$. **d)** $x^2 + x + 1$ is not $o(x^2)$.

62. (*Requires calculus*)

a) Show that if $f(x)$ and $g(x)$ are functions such that $f(x)$ is $o(g(x))$ and c is a constant, then $cf(x)$ is $o(g(x))$, where $(cf)(x) = cf(x)$.

b) Show that if $f_1(x)$, $f_2(x)$, and $g(x)$ are functions such that $f_1(x)$ is $o(g(x))$ and $f_2(x)$ is $o(g(x))$, then $(f_1 + f_2)(x)$ is $o(g(x))$, where $(f_1 + f_2)(x) = f_1(x) + f_2(x)$.

63. (*Requires calculus*) Represent pictorially that $x \log x$ is $o(x^2)$ by graphing $x \log x$, x^2, and $x \log x / x^2$. Explain how this picture shows that $x \log x$ is $o(x^2)$.

64. (*Requires calculus*) Express the relationship $f(x)$ is $o(g(x))$ using a picture. Show the graphs of $f(x)$, $g(x)$, and $f(x)/g(x)$.

***65.** (*Requires calculus*) Suppose that $f(x)$ is $o(g(x))$. Does it follow that $2^{f(x)}$ is $o(2^{g(x)})$?

***66.** (*Requires calculus*) Suppose that $f(x)$ is $o(g(x))$. Does it follow that $\log |f(x)|$ is $o(\log |g(x)|)$?

67. (*Requires calculus*) The two parts of this exercise describe the relationship between little-o and big-O notation.

a) Show that if $f(x)$ and $g(x)$ are functions such that $f(x)$ is $o(g(x))$, then $f(x)$ is $O(g(x))$.

b) Show that if $f(x)$ and $g(x)$ are functions such that $f(x)$ is $O(g(x))$, then it does not necessarily follow that $f(x)$ is $o(g(x))$.

68. (*Requires calculus*) Show that if $f(x)$ is a polynomial of degree n and $g(x)$ is a polynomial of degree m where $m > n$, then $f(x)$ is $o(g(x))$.

69. (*Requires calculus*) Show that if $f_1(x)$ is $O(g(x))$ and $f_2(x)$ is $o(g(x))$, then $f_1(x) + f_2(x)$ is $O(g(x))$.

70. (*Requires calculus*) Let H_n be the nth **harmonic number**

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Show that H_n is $O(\log n)$. [*Hint:* First establish the inequality

$$\sum_{j=2}^{n} \frac{1}{j} < \int_1^n \frac{1}{x} dx$$

by showing that the sum of the areas of the rectangles of height $1/j$ with base from $j - 1$ to j, for $j = 2, 3, \ldots, n$, is less than the area under the curve $y = 1/x$ from 2 to n.]

***71.** Show that $n \log n$ is $O(\log n!)$.

72. Determine whether $\log n!$ is $\Theta(n \log n)$. Justify your answer.

***73.** Show that $\log n!$ is greater than $(n \log n)/4$ for $n > 4$. [*Hint:* Begin with the inequality $n! > n(n-1)(n-2) \cdots \lceil n/2 \rceil$.]

Let $f(x)$ and $g(x)$ be functions from the set of real numbers to the set of real numbers. We say that the functions f and g are **asymptotic** and write $f(x) \sim g(x)$ if $\lim_{x\to\infty} f(x)/g(x) = 1$.

74. (*Requires calculus*) For each of these pairs of functions, determine whether f and g are asymptotic.

a) $f(x) = x^2 + 3x + 7$, $g(x) = x^2 + 10$

b) $f(x) = x^2 \log x$, $g(x) = x^3$

c) $f(x) = x^4 + \log(3x^8 + 7)$, $g(x) = (x^2 + 17x + 3)^2$

d) $f(x) = (x^3 + x^2 + x + 1)^4$, $g(x) = (x^4 + x^3 + x^2 + x + 1)^3$.

75. (*Requires calculus*) For each of these pairs of functions, determine whether f and g are asymptotic.

a) $f(x) = \log(x^2 + 1)$, $g(x) = \log x$

b) $f(x) = 2^{x+3}$, $g(x) = 2^{x+7}$

c) $f(x) = 2^{2^x}$, $g(x) = 2^{x^2}$

d) $f(x) = 2^{x^2+x+1}$, $g(x) = 2^{x^2+2x}$

3.3 Complexity of Algorithms

Introduction

When does an algorithm provide a satisfactory solution to a problem? First, it must always produce the correct answer. How this can be demonstrated will be discussed in Chapter 5. Second, it should be efficient. The efficiency of algorithms will be discussed in this section.

How can the efficiency of an algorithm be analyzed? One measure of efficiency is the time used by a computer to solve a problem using the algorithm, when input values are of a specified