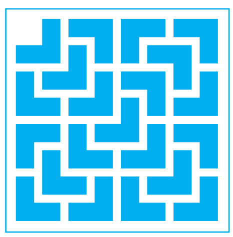
3 are relatively prime to 10. Therefore the sum can no longer be 0 modulo 10. **45.** Working modulo 10, solve for d_9. The check digit for 11100002 is 5. **47.** PLEASE SEND MONEY **49. a)** QAL HUVEM AT WVESGB **b)** QXB EVZZL ZEVZZRFS

CHAPTER 5

Section 5.1

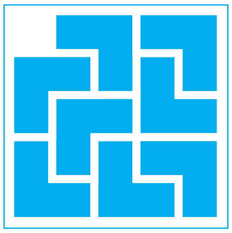
1. Let $P(n)$ be the statement that the train stops at station n. *Basis step:* We are told that $P(1)$ is true. *Inductive step:* We are told that $P(n)$ implies $P(n+1)$ for each $n \geq 1$. Therefore by the principle of mathematical induction, $P(n)$ is true for all positive integers n. **3. a)** $1^2 = 1 \cdot 2 \cdot 3/6$ **b)** Both sides of $P(1)$ shown in part (a) equal 1. **c)** $1^2 + 2^2 + \cdots + k^2 = k(k+1)(2k+1)/6$ **d)** For each $k \geq 1$ that $P(k)$ implies $P(k+1)$; in other words, that assuming the inductive hypothesis [see part (c)] we can show $1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = (k+1)(k+2)(2k+3)/6$ **e)** $(1^2 + 2^2 + \cdots + k^2) + (k+1)^2 = [k(k+1)(2k+1)/6] + (k+1)^2 = [(k+1)/6][k(2k+1) + 6(k+1)] = [(k+1)/6](2k^2 + 7k + 6) = [(k+1)/6](k+2)(2k+3) = (k+1)(k+2)(2k+3)/6$ **f)** We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n. **5.** Let $P(n)$ be "$1^2 + 3^2 + \cdots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$." *Basis step:* $P(0)$ is true because $1^2 = 1 = (0+1)(2 \cdot 0+1)(2 \cdot 0+3)/3$. *Inductive step:* Assume that $P(k)$ is true. Then $1^2 + 3^2 + \cdots + (2k+1)^2 + [2(k+1)+1]^2 = (k+1)(2k+1)(2k+3)/3 + (2k+3)^2 = (2k+3)[(k+1)(2k+1)/3 + (2k+3)] = (2k+3)(2k^2+9k+10)/3 = (2k+3)(2k+5)(k+2)/3 = [(k+1)+1][2(k+1)+1][2(k+1)+3]/3$. **7.** Let $P(n)$ be "$\sum_{j=0}^{n} 3 \cdot 5^j = 3(5^{n+1} - 1)/4$." *Basis step:* $P(0)$ is true because $\sum_{j=0}^{0} 3 \cdot 5^j = 3 = 3(5^1 - 1)/4$. *Inductive step:* Assume that $\sum_{j=0}^{k} 3 \cdot 5^j = 3(5^{k+1} - 1)/4$. Then $\sum_{j=0}^{k+1} 3 \cdot 5^j = (\sum_{j=0}^{k} 3 \cdot 5^j) + 3 \cdot 5^{k+1} = 3(5^{k+1} - 1)/4 + 3 \cdot 5^{k+1} = 3(5^{k+1} + 4 \cdot 5^{k+1} - 1)/4 = 3(5^{k+2} - 1)/4$. **9. a)** $2+4+6+\cdots+2n = n(n+1)$ **b)** *Basis step:* $2 = 1 \cdot (1+1)$ is true. *Inductive step:* Assume that $2 + 4 + 6 + \cdots + 2k = k(k+1)$. Then $(2 + 4 + 6 + \cdots + 2k) + 2(k+1) = k(k+1) + 2(k+1) = (k+1)(k+2)$. **11. a)** $\sum_{j=1}^{n} 1/2^j = (2^n - 1)/2^n$ **b)** *Basis step:* $P(1)$ is true because $\frac{1}{2} = (2^1 - 1)/2^1$. *Inductive step:* Assume that $\sum_{j=1}^{k} 1/2^j = (2^k - 1)/2^k$. Then $\sum_{j=1}^{k+1} \frac{1}{2^j} = (\sum_{j=1}^{k} \frac{1}{2^j}) + \frac{1}{2^{k+1}} = \frac{2^k - 1}{2^k} + \frac{1}{2^{k+1}} = \frac{2^{k+1} - 2 + 1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}}$. **13.** Let $P(n)$ be "$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}n(n+1)/2$." *Basis step:* $P(1)$ is true because $1^2 = 1 = (-1)^0 1^2$. *Inductive step:* Assume that $P(k)$ is true. Then $1^2 - 2^2 + 3^2 - \cdots + (-1)^{k-1}k^2 + (-1)^k(k+1)^2 = (-1)^{k-1}k(k+1)/2 + (-1)^k(k+1)^2 = (-1)^k(k+1)[-k/2 + (k+1)] = (-1)^k(k+1)[(k/2)+1] = (-1)^k(k+1)(k+2)/2$. **15.** Let $P(n)$ be "$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = n(n+1)(n+2)/3$." *Basis step:* $P(1)$ is true because $1 \cdot 2 = 2 = 1(1+1)(1+2)/3$. *Inductive step:* Assume that $P(k)$ is true. Then $1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) + (k+1)(k+2) = [k(k+1)(k+2)/3] + (k+1)(k+2) = (k+1)(k+2)[(k/3)+1] = (k+1)(k+2)(k+3)/3$. **17.** Let $P(n)$ be the statement that $1^4 + 2^4 + 3^4 + \cdots + n^4 = n(n+1)(2n+1)(3n^2+3n-1)/30$. $P(1)$ is true because $1 \cdot 2 \cdot 3 \cdot 5/30 = 1$. Assume that $P(k)$ is true. Then $(1^4 + 2^4 + 3^4 + \cdots + k^4) + (k+1)^4 = k(k+1)(2k+1)(3k^2 + 3k - 1)/30 + (k+1)^4 = [(k+1)/30][k(2k+1)(3k^2 + 3k - 1) + 30(k+1)^3] = [(k+1)/30](6k^4 + 39k^3 + 91k^2 + 89k + 30) = [(k+1)/30](k+2)(2k+3)[3(k+1)^2 + 3(k+1) - 1]$. This demonstrates that $P(k+1)$ is true. **19. a)** $1 + \frac{1}{4} < 2 - \frac{1}{2}$ **b)** This is true because $5/4$ is less than $6/4$. **c)** $1 + \frac{1}{4} + \cdots + \frac{1}{k^2} < 2 - \frac{1}{k}$ **d)** For each $k \geq 2$ that $P(k)$ implies $P(k+1)$; in other words, we want to show that assuming the inductive hypothesis [see part (c)] we can show $1 + \frac{1}{4} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$ **e)** $1 + \frac{1}{4} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k} + \frac{1}{(k+1)^2} = 2 - \left[\frac{1}{k} - \frac{1}{(k+1)^2}\right] = 2 - \left[\frac{k^2+2k+1-k}{k(k+1)^2}\right] = 2 - \frac{k^2+k}{k(k+1)^2} - \frac{1}{k(k+1)^2} = 2 - \frac{1}{k+1} - \frac{1}{k(k+1)^2} < 2 - \frac{1}{k+1}$ **f)** We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every integer n greater than 1. **21.** Let $P(n)$ be "$2^n > n^2$." *Basis step:* $P(5)$ is true because $2^5 = 32 > 25 = 5^2$. *Inductive step:* Assume that $P(k)$ is true, that is, $2^k > k^2$. Then $2^{k+1} = 2 \cdot 2^k > k^2 + k^2 > k^2 + 4k \geq k^2 + 2k + 1 = (k+1)^2$ because $k > 4$. **23.** By inspection we find that the inequality $2n + 3 \leq 2^n$ does not hold for $n = 0, 1, 2, 3$. Let $P(n)$ be the proposition that this inequality holds for the positive integer n. $P(4)$, the basis case, is true because $2 \cdot 4 + 3 = 11 \leq 16 = 2^4$. For the inductive step assume that $P(k)$ is true. Then, by the inductive hypothesis, $2(k+1)+3 = (2k+3)+2 < 2^k+2$. But because $k \geq 1$, $2^k+2 \leq 2^k+2^k = 2^{k+1}$. This shows that $P(k+1)$ is true. **25.** Let $P(n)$ be "$1 + nh \leq (1+h)^n, h > -1$." *Basis step:* $P(0)$ is true because $1 + 0 \cdot h = 1 \leq 1 = (1+h)^0$. *Inductive step:* Assume $1 + kh \leq (1+h)^k$. Then because $(1+h) > 0$, $(1+h)^{k+1} = (1+h)(1+h)^k \geq (1+h)(1+kh) = 1 + (k+1)h + kh^2 \geq 1 + (k+1)h$. **27.** Let $P(n)$ be "$1/\sqrt{1} + 1/\sqrt{2} + 1/\sqrt{3} + \cdots + 1/\sqrt{n} > 2\left(\sqrt{n+1} - 1\right)$." *Basis step:* $P(1)$ is true because $1 > 2\left(\sqrt{2} - 1\right)$. *Inductive step:* Assume that $P(k)$ is true. Then $1 + 1/\sqrt{2} + \cdots + 1/\sqrt{k} + 1/\sqrt{k+1} > 2\left(\sqrt{k+1} - 1\right) + 1/\sqrt{k+1}$. If we show that $2\left(\sqrt{k+1} - 1\right) + 1/\sqrt{k+1} > 2\left(\sqrt{k+2} - 1\right)$, it follows that $P(k+1)$ is true. This inequality is equivalent to $2\left(\sqrt{k+2} - \sqrt{k+1}\right) < 1/\sqrt{k+1}$, which is equivalent to $2\left(\sqrt{k+2} - \sqrt{k+1}\right)\left(\sqrt{k+2} + \sqrt{k+1}\right) < \sqrt{k+1}/\sqrt{k+1} + \sqrt{k+2}/\sqrt{k+1}$. This is equivalent to $2 < 1 + \sqrt{k+2}/\sqrt{k+1}$, which is clearly true. **29.** Let $P(n)$ be "$H_{2^n} \leq 1 + n$." *Basis step:* $P(0)$ is true because $H_{2^0} = H_1 = 1 \leq 1 + 0$. *Inductive step:* Assume that $H_{2^k} \leq 1 + k$. Then $H_{2^{k+1}} = H_{2^k} + \sum_{j=2^k+1}^{2^{k+1}} \frac{1}{j} \leq 1 + k + 2^k\left(\frac{1}{2^{k+1}}\right) < 1 + k + 1 = 1 + (k+1)$. **31.** *Basis step:* $1^2 + 1 = 2$ is divisible by 2. *Inductive step:* Assume the inductive hypothesis, that $k^2 + k$ is divisible by 2. Then $(k+1)^2 + (k+1) = k^2 + 2k + 1 + k + 1 = (k^2 + k) + 2(k+1)$, the sum of a multiple of 2 (by the inductive hypothesis) and a multiple of 2 (by definition), hence, divisible by 2. **33.** Let $P(n)$ be "$n^5 - n$ is divisible by 5." *Basis step:* $P(0)$ is true because $0^5 - 0 = 0$ is divisible by 5. *Inductive step:* Assume that $P(k)$ is true, that is, $k^5 - 5$ is divisible by 5. Then $(k+1)^5-(k+1) = (k^5+5k^4+10k^3+10k^2+5k+1)-(k+1) = (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k)$ is also divisible by 5, because both terms in this sum are divisible by 5. **35.** Let $P(n)$ be the proposition that $(2n - 1)^2 - 1$ is divisible by 8. The basis case $P(1)$ is true because $8 \mid 0$. Now assume that $P(k)$ is true. Because $[(2(k+1) - 1]^2 - 1 = [(2k-1)^2 - 1] + 8k$, $P(k+1)$ is true because both terms on the right-hand side are divisible by 8. This shows that $P(n)$ is true for all positive integers n, so $m^2 - 1$ is divisible by 8 whenever m is an odd positive integer. **37.** *Basis step:* $11^{1+1}+12^{2\cdot 1-1} = 121+12 = 133$ *Inductive step:* Assume the inductive hypothesis, that $11^{n+1} + 12^{2n-1}$ is divisible by 133. Then $11^{(n+1)+1} + 12^{2(n+1)-1} = 11 \cdot 11^{n+1} + 144 \cdot 12^{2n-1} = 11 \cdot 11^{n+1} + (11+133) \cdot 12^{2n-1} = 11(11^{n+1} + 12^{2n-1}) + 133 \cdot 12^{2n-1}$. The expression in parentheses is divisible by 133 by the inductive hypothesis, and obviously the second term is divisible by 133, so the entire quantity is divisible by 133, as desired. **39.** *Basis step:* $A_1 \subseteq B_1$ tautologically implies that $\bigcap_{j=1}^{1} A_j \subseteq \bigcap_{j=1}^{1} B_j$. *Inductive step:* Assume the inductive hypothesis that if $A_j \subseteq B_j$ for $j = 1, 2, \ldots, k$, then $\bigcap_{j=1}^{k} A_j \subseteq \bigcap_{j=1}^{k} B_j$. We want to show that if $A_j \subseteq B_j$ for $j = 1, 2, \ldots, k+1$, then $\bigcap_{j=1}^{k+1} A_j \subseteq \bigcap_{j=1}^{k+1} B_j$. Let x be an arbitrary element of $\bigcap_{j=1}^{k+1} A_j = \left(\bigcap_{j=1}^{k} A_j\right) \cap A_{k+1}$. Because $x \in \bigcap_{j=1}^{k} A_j$, we know by the inductive hypothesis that $x \in \bigcap_{j=1}^{k} B_j$; because $x \in A_{k+1}$, we know from the given fact that $A_{k+1} \subseteq B_{k+1}$ that $x \in B_{k+1}$. Therefore, $x \in \left(\bigcap_{j=1}^{k} B_j\right) \cap B_{k+1} = \bigcap_{j=1}^{k+1} B_j$. **41.** Let $P(n)$ be "$(A_1 \cup A_2 \cup \cdots \cup A_n) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)$." *Basis step:* $P(1)$ is trivially true. *Inductive step:* Assume that $P(k)$ is true. Then $(A_1 \cup A_2 \cup \cdots \cup A_k \cup A_{k+1}) \cap B = [(A_1 \cup A_2 \cup \cdots \cup A_k) \cup A_{k+1}] \cap B = [(A_1 \cup A_2 \cup \cdots \cup A_k) \cap B] \cup (A_{k+1} \cap B) = [(A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_k \cap B)] \cup (A_{k+1} \cap B) = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_k \cap B) \cup (A_{k+1} \cap B)$. **43.** Let $P(n)$ be "$\overline{\bigcup_{k=1}^{n} A_k} = \bigcap_{k=1}^{n} \overline{A_k}$." *Basis step:* $P(1)$ is trivially true. *Inductive step:* Assume that $P(k)$ is true. Then $\overline{\bigcup_{j=1}^{k+1} A_j} = \overline{\left(\bigcup_{j=1}^{k} A_j\right) \cup A_{k+1}} = \overline{\left(\bigcup_{j=1}^{k} A_j\right)} \cap \overline{A_{k+1}} = \left(\bigcap_{j=1}^{k} \overline{A_j}\right) \cap \overline{A_{k+1}} = \bigcap_{j=1}^{k+1} \overline{A_j}$. **45.** Let $P(n)$ be the statement that a set with n elements has $n(n-1)/2$ two-element subsets. $P(2)$, the basis case, is true, because a set with two elements has one subset with two elements—namely, itself—and $2(2-1)/2 = 1$. Now assume that $P(k)$ is true. Let S be a set with $k+1$ elements. Choose an element a in S and let $T = S - \{a\}$. A two-element subset of S either contains a or does not. Those subsets not containing a are the subsets of T with two elements; by the inductive hypothesis there are $k(k-1)/2$ of these. There are k subsets of S with two elements that contain a, because such a subset contains a and one of the k elements in T. Hence, there are $k(k-1)/2+k = (k+1)k/2$ two-element subsets of S. This completes the inductive proof. **47.** Reorder the locations if necessary so that $x_1 \le x_2 \le x_3 \le \cdots \le x_d$. Place the first tower at position $t_1 = x_1 + 1$. Assume tower k has been placed at position t_k. Then place tower $k+1$ at position $t_{k+1} = x + 1$, where x is the smallest x_i greater than $t_k + 1$. **49.** The two sets do not overlap if $n+1 = 2$. In fact, the conditional statement $P(1) \to P(2)$ is false. **51.** The mistake is in applying the inductive hypothesis to look at $\max(x-1, y-1)$, because even though x and y are positive integers, $x-1$ and $y-1$ need not be (one or both could be 0). **53.** For the basis step ($n = 2$) the first person cuts the cake into two portions that she thinks are each $1/2$ of the cake, and the second person chooses the portion he thinks is at least $1/2$ of the cake (at least one of the pieces must satisfy that condition). For the inductive step, suppose there are $k+1$ people. By the inductive hypothesis, we can suppose that the first k people have divided the cake among themselves so that each person is satisfied that he got at least a fraction $1/k$ of the cake. Each of them now cuts his or her piece into $k+1$ pieces of equal size. The last person gets to choose one piece from each of the first k people's portions. After this is done, each of the first k people is satisfied that she still has $(1/k)(k/(k+1)) = 1/(k+1)$ of the cake. To see that the last person is satisfied, suppose that he thought that the ith person ($1 \le i \le k$) had a portion p_i of the cake, where $\sum_{i=1}^{k} p_i = 1$. By choosing what he thinks is the largest piece from each person, he is satisfied that he has at least $\sum_{i=1}^{k} p_i/(k+1) = (1/(k+1)) \sum_{i=1}^{k} p_i = 1/(k+1)$ of the cake. **55.** We use the notation (i, j) to mean the square in row i and column j and use induction on $i + j$ to show that every square can be reached by the knight. *Basis step:* There are six base cases, for the cases when $i + j \le 2$. The knight is already at $(0, 0)$ to start, so the empty sequence of moves reaches that square. To reach $(1, 0)$, the knight moves $(0, 0) \to (2, 1) \to (0, 2) \to (1, 0)$. Similarly, to reach $(0, 1)$, the knight moves $(0, 0) \to (1, 2) \to (2, 0) \to (0, 1)$. Note that the knight has reached $(2, 0)$ and $(0, 2)$ in the process. For the last basis step there is $(0, 0) \to (1, 2) \to (2, 0) \to (0, 1) \to (2, 2) \to (0, 3) \to (1, 1)$. *Inductive step:* Assume the inductive hypothesis, that the knight can reach any square (i, j) for which $i + j = k$, where k is an integer greater than 1. We must show how the knight can reach each square (i, j) when $i + j = k + 1$. Because $k + 1 \ge 3$, at least one of i and j is at least 2. If $i \ge 2$, then by the inductive hypothesis, there is a sequence of moves ending at $(i-2, j+1)$, because $i - 2 + j + 1 = i + j - 1 = k$; from there it is just one step to (i, j); similarly, if $j \ge 2$. **57.** *Basis step:* The base cases $n = 0$ and $n = 1$ are true because the derivative of x^0 is 0 and the derivative of $x^1 = x$ is 1. *Inductive step:* Using the product rule, the inductive hypothesis, and the basis step shows that $\frac{d}{dx}x^{k+1} = \frac{d}{dx}(x \cdot x^k) = x \cdot \frac{d}{dx}x^k + x^k \frac{d}{dx}x = x \cdot kx^{k-1} + x^k \cdot 1 = kx^k + x^k = (k+1)x^k$. **59.** *Basis step:* For $k = 0$, $1 \equiv 1 \pmod{m}$. *Inductive step:* Suppose that $a \equiv b \pmod{m}$ and $a^k \equiv b^k \pmod{m}$; we must show that $a^{k+1} \equiv b^{k+1} \pmod{m}$. By Theorem 5 from Section 4.1, $a \cdot a^k \equiv b \cdot b^k \pmod{m}$, which by definition says that $a^{k+1} \equiv b^{k+1} \pmod{m}$. **61.** Let $P(n)$ be "$[(p_1 \to p_2) \wedge (p_2 \to p_3) \wedge \cdots \wedge (p_{n-1} \to p_n)] \to [(p_1 \wedge \cdots \wedge p_{n-1}) \to p_n]$." *Basis step:* $P(2)$ is true because $(p_1 \to p_2) \to (p_1 \to p_2)$ is a tautology. *Inductive step:* Assume $P(k)$ is true. To show $[(p_1 \to p_2) \wedge \cdots \wedge (p_{k-1} \to p_k) \wedge (p_k \to p_{k+1})] \to [(p_1 \wedge \cdots \wedge p_{k-1} \wedge p_k) \to p_{k+1}]$ is a tautology, assume that the hypothesis of this conditional statement is true. Because both the hypothesis and $P(k)$ are true, it follows that $(p_1 \wedge \cdots \wedge p_{k-1}) \to p_k$ is true. Because this is true, and because $p_k \to p_{k+1}$ is true (it is part of the assumption) it follows by hypothetical syllogism that $(p_1 \wedge \cdots \wedge p_{k-1}) \to p_{k+1}$ is true. The weaker statement $(p_1 \wedge \cdots \wedge p_{k-1} \wedge p_k) \to p_{k+1}$ follows from this. **63.** We will first prove the result when n is a power of 2, that is, if $n = 2^k$, $k = 1, 2, \ldots$. Let $P(k)$ be the statement $A \geq G$, where A and G are the arithmetic and geometric means, respectively, of a set of $n = 2^k$ positive real numbers. *Basis step:* $k = 1$ and $n = 2^1 = 2$. Note that $(\sqrt{a_1} - \sqrt{a_2})^2 \geq 0$. Expanding this shows that $a_1 - 2\sqrt{a_1 a_2} + a_2 \geq 0$, that is, $(a_1 + a_2)/2 \geq (a_1 a_2)^{1/2}$. *Inductive step:* Assume that $P(k)$ is true, with $n = 2^k$. We will show that $P(k+1)$ is true. We have $2^{k+1} = 2n$. Now $(a_1 + a_2 + \cdots + a_{2n})/(2n) = [(a_1 + a_2 + \cdots + a_n)/n + (a_{n+1} + a_{n+2} + \cdots + a_{2n})/n]/2$ and similarly $(a_1 a_2 \cdots a_{2n})^{1/(2n)} = [(a_1 \cdots a_n)^{1/n} (a_{n+1} \cdots a_{2n})^{1/n}]^{1/2}$. To simplify the notation, let $A(x, y, \ldots)$ and $G(x, y, \ldots)$ denote the arithmetic mean and geometric mean of $x, y, \ldots$, respectively. Also, if $x \leq x'$, $y \leq y'$, and so on, then $A(x, y, \ldots) \leq A(x', y', \ldots)$ and $G(x, y, \ldots) \leq G(x', y', \ldots)$. Hence, $A(a_1, \ldots, a_{2n}) = A(A(a_1, \ldots, a_n), A(a_{n+1}, \ldots, a_{2n})) \geq A(G(a_1, \ldots, a_n), G(a_{n+1}, \ldots, a_{2n})) \geq G(G(a_1, \ldots, a_n), G(a_{n+1}, \ldots, a_{2n})) = G(a_1, \ldots, a_{2n})$. This finishes the proof for powers of 2. Now if n is not a power of 2, let m be the next higher power of 2, and let $a_{n+1}, \ldots, a_m$ all equal $A(a_1, \ldots, a_n) = \overline{a}$. Then we have $[(a_1 a_2 \cdots a_n)\overline{a}^{m-n}]^{1/m} \leq A(a_1, \ldots, a_m)$, because m is a power of 2. Because $A(a_1, \ldots, a_m) = \overline{a}$, it follows that $(a_1 \cdots a_n)^{1/m} \overline{a}^{1-n/m} \leq \overline{a}^{n/m}$. Raising both sides to the (m/n)th power gives $G(a_1, \ldots, a_n) \leq A(a_1, \ldots, a_n)$. **65.** *Basis step:* For $n = 1$, the left-hand side is just $\frac{1}{1}$, which is 1. For $n = 2$, there are three nonempty subsets $\{1\}$, $\{2\}$, and $\{1, 2\}$, so the left-hand side is $\frac{1}{1} + \frac{1}{2} + \frac{1}{1 \cdot 2} = 2$. *Inductive step:* Assume that the statement is true for k. The set of the first $k + 1$ positive integers has many nonempty subsets, but they fall into three categories: a nonempty subset of the first k positive integers together with $k + 1$, a nonempty subset of the first k positive integers, or just $\{k + 1\}$. By the inductive hypothesis, the sum of the first category is k. For the second category, we can factor out $1/(k + 1)$ from each term of the sum and what remains is just k by the inductive hypothesis, so this part of the sum is $k/(k + 1)$. Finally, the third category simply yields $1/(k + 1)$. Hence, the entire summation is $k + k/(k + 1) + 1/(k + 1) = k + 1$. **67.** *Basis step:* If $A_1 \subseteq A_2$, then A_1 satisfies the condition of being a subset of each set in the collection; otherwise $A_2 \subseteq A_1$, so A_2 satisfies the condition. *Inductive step:* Assume the inductive hypothesis, that the conditional statement is true for k sets, and suppose we are given $k + 1$ sets that satisfy the given conditions. By the inductive hypothesis, there must be a set A_i for some $i \leq k$ such that $A_i \subseteq A_j$ for $1 \leq j \leq k$. If $A_i \subseteq A_{k+1}$, then we are done. Otherwise, we know that $A_{k+1} \subseteq A_i$, and this tells us that A_{k+1} satisfies the condition of being a subset of A_j for $1 \leq j \leq k + 1$. **69.** $G(1) = 0$, $G(2) = 1$, $G(3) = 3$, $G(4) = 4$ **71.** To show that $2n - 4$ calls are sufficient to exchange all the gossip, select persons 1, 2, 3, and 4 to be the central committee. Every person outside the central committee calls one person on the central committee. At this point the central committee members *as a group* know all the scandals. They then exchange information among themselves by making the calls 1-2, 3-4, 1-3, and 2-4 in that order. At this point, *every* central committee member knows all the scandals. Finally, again every person outside the central committee calls one person on the central committee, at which point everyone knows all the scandals. [The total number of calls is $(n - 4) + 4 + (n - 4) = 2n - 4$.] That this cannot be done with fewer than $2n - 4$ calls is much harder to prove; see Sandra M. Hedetniemi, Stephen T. Hedetniemi, and Arthur L. Liestman, "A survey of gossiping and broadcasting in communication networks," *Networks* **18** (1988), no. 4, 319–349, for details. **73.** We prove this by mathematical induction. The basis step ($n = 2$) is true tautologically. For $n = 3$, suppose that the intervals are (a, b), (c, d), and (e, f), where without loss of generality we can assume that $a \leq c \leq e$. Because $(a, b) \cap (e, f) \neq \emptyset$, we must have $e < b$; for a similar reason, $e < d$. It follows that the number halfway between e and the smaller of b and d is common to all three intervals. Now for the inductive step, assume that whenever we have k intervals that have pairwise nonempty intersections then there is a point common to all the intervals, and suppose that we are given intervals $I_1, I_2, \ldots, I_{k+1}$ that have pairwise nonempty intersections. For each i from 1 to k, let $J_i = I_i \cap I_{k+1}$. We claim that the collection $J_1, J_2, \ldots, J_k$ satisfies the inductive hypothesis, that is, that $J_{i_1} \cap J_{i_2} \neq \emptyset$ for each choice of subscripts i_1 and i_2. This follows from the $n = 3$ case proved above, using the sets I_{i_1}, I_{i_2}, and I_{k+1}. We can now invoke the inductive hypothesis to conclude that there is a number common to all of the sets J_i for $i = 1, 2, \ldots, k$, which perforce is in the intersection of all the sets I_i for $i = 1, 2, \ldots, k + 1$. **75.** Pair up the people. Have the people stand at mutually distinct small distances from their partners but far away from everyone else. Then each person throws a pie at his or her partner, so everyone gets hit.

77.



79. Let $P(n)$ be the statement that every $2^n \times 2^n \times 2^n$ checkerboard with a $1 \times 1 \times 1$ cube removed can be covered by tiles that are $2 \times 2 \times 2$ cubes each with a $1 \times 1 \times 1$ cube removed. The basis step, $P(1)$, holds because one tile coincides with the solid to be tiled. Now assume that $P(k)$ holds. Now consider a $2^{k+1} \times 2^{k+1} \times 2^{k+1}$ cube with a $1 \times 1 \times 1$ cube removed. Split this object into eight pieces using planes parallel to its faces and running through its center. The missing $1 \times 1 \times 1$ piece occurs in one of these eight pieces. Now position one tile with its center at the center of the large object so that the missing $1 \times 1 \times 1$ cube lies in the octant in which the large object is missing a $1 \times 1 \times 1$ cube. This creates eight $2^k \times 2^k \times 2^k$ cubes, each missing a $1 \times 1 \times 1$ cube. By the inductive hypothesis we can fill each of these eight objects with tiles. Putting these tilings together produces the desired tiling.

81.



83. Let $Q(n)$ be $P(n+b-1)$. The statement that $P(n)$ is true for $n = b, b+1, b+2, \ldots$ is the same as the statement that $Q(m)$ is true for all positive integers m. We are given that $P(b)$ is true [i.e., that $Q(1)$ is true], and that $P(k) \rightarrow P(k+1)$ for all $k \geq b$ [i.e., that $Q(m) \rightarrow Q(m+1)$ for all positive integers m]. Therefore, by the principle of mathematical induction, $Q(m)$ is true for all positive integers m.

Section 5.2

1. *Basis step:* We are told we can run one mile, so $P(1)$ is true. *Inductive step:* Assume the inductive hypothesis, that we can run any number of miles from 1 to k. We must show that we can run $k+1$ miles. If $k = 1$, then we are already told that we can run two miles. If $k > 1$, then the inductive hypothesis tells us that we can run $k-1$ miles, so we can run $(k-1)+2 = k+1$ miles. **3. a)** $P(8)$ is true, because we can form 8 cents of postage with one 3-cent stamp and one 5-cent stamp. $P(9)$ is true, because we can form 9 cents of postage with three 3-cent stamps. $P(10)$ is true, because we can form 10 cents of postage with two 5-cent stamps. **b)** The statement that using just 3-cent and 5-cent stamps we can form j cents postage for all j with $8 \leq j \leq k$, where we assume that $k \geq 10$ **c)** Assuming the inductive hypothesis, we can form $k+1$ cents postage using just 3-cent and 5-cent stamps **d)** Because $k \geq 10$, we know that $P(k-2)$ is true, that is, that we can form $k-2$ cents of postage. Put one more 3-cent stamp on the envelope, and we have formed $k+1$ cents of postage. **e)** We have completed both the basis step and the inductive step, so by the principle of strong induction, the statement is true for every integer n greater than or equal to 8. **5. a)** 4, 8, 11, 12, 15, 16, 19, 20, 22, 23, 24, 26, 27, 28, and all values greater than or equal to 30 **b)** Let $P(n)$ be the statement that we can form n cents of postage using just 4-cent and 11-cent stamps. We want to prove that $P(n)$ is true for all $n \geq 30$. For the basis step, $30 = 11+11+4+4$. Assume that we can form k cents of postage (the inductive hypothesis); we will show how to form $k+1$ cents of postage. If the k cents included an 11-cent stamp, then replace it by three 4-cent stamps. Otherwise, k cents was formed from just 4-cent stamps. Because $k \geq 30$, there must be at least eight 4-cent stamps involved. Replace eight 4-cent stamps by three 11-cent stamps, and we have formed $k+1$ cents in postage. **c)** $P(n)$ is the same as in part (b). To prove that $P(n)$ is true for all $n \geq 30$, we check for the basis step that $30 = 11+11+4+4$, $31 = 11+4+4+4+4+4$, $32 = 4+4+4+4+4+4+4+4$, and $33 = 11+11+11$. For the inductive step, assume the inductive hypothesis, that $P(j)$ is true for all j with $30 \leq j \leq k$, where k is an arbitrary integer greater than or equal to 33. We want to show that $P(k+1)$ is true. Because $k-3 \geq 30$, we know that $P(k-3)$ is true, that is, that we can form $k-3$ cents of postage. Put one more 4-cent stamp on the envelope, and we have formed $k+1$ cents of postage. In this proof, our inductive hypothesis was that $P(j)$ was true for all values of j between 30 and k inclusive, rather than just that $P(30)$ was true. **7.** We can form all amounts except \$1 and \$3. Let $P(n)$ be the statement that we can form n dollars using just 2-dollar and 5-dollar bills. We want to prove that $P(n)$ is true for all $n \geq 5$. (It is clear that \$1 and \$3 cannot be formed and that \$2 and \$4 can be formed.) For the basis step, note that $5 = 5$ and $6 = 2+2+2$. Assume the inductive hypothesis, that $P(j)$ is true for all j with $5 \leq j \leq k$, where k is an arbitrary integer greater than or equal to 6. We want to show that $P(k+1)$ is true. Because $k-1 \geq 5$, we know that $P(k-1)$ is true, that is, that we can form $k-1$ dollars. Add another 2-dollar bill, and we have formed $k+1$ dollars. **9.** Let $P(n)$ be the statement that there is no positive integer b such that $\sqrt{2} = n/b$. *Basis step:* $P(1)$ is true because $\sqrt{2} > 1 \geq 1/b$ for all positive integers b. *Inductive step:* Assume that $P(j)$ is true for all $j \leq k$, where k is an arbitrary positive integer; we prove that $P(k+1)$ is true by contradiction. Assume that $\sqrt{2} = (k+1)/b$ for some positive integer b. Then $2b^2 = (k+1)^2$, so $(k+1)^2$ is even, and hence, $k+1$ is even. So write $k+1 = 2t$ for some positive integer t, whence $2b^2 = 4t^2$ and $b^2 = 2t^2$. By the same reasoning as before, b is even, so $b = 2s$ for some positive integer s. Then $\sqrt{2} = (k+1)/b = (2t)/(2s) = t/s$. But $t \leq k$, so this contradicts the inductive hypothesis, and our proof of the inductive step is complete. **11.** *Basis step:* There are four base cases. If $n = 1 = 4 \cdot 0 + 1$, then clearly the second player wins. If there are two, three, or four matches ($n = 4 \cdot 0 + 2$, $n = 4 \cdot 0 + 3$, or $n = 4 \cdot 1$), then the first player can win by removing all but one match. *Inductive step:* Assume the strong inductive hypothesis, that in games with k or fewer matches, the first player can win if $k \equiv 0, 2$, or $3 \pmod 4$ and the second player can win if $k \equiv 1 \pmod 4$. Suppose we have a game with $k+1$ matches, with $k \geq 4$. If $k+1 \equiv 0 \pmod 4$, then the first player can remove three matches, leaving $k-2$ matches for the other player. Because $k-2 \equiv 1 \pmod 4$, by the inductive hypothesis, this is a game that the second player at that point (who is the first player in our game) can win. Similarly, if $k+1 \equiv 2 \pmod 4$, then the first player can remove one match; and if $k+1 \equiv 3 \pmod 4$, then the first player can remove two matches. Finally, if $k+1 \equiv 1 \pmod 4$, then the first player must leave k, $k-1$, or $k-2$ matches for the other player. Because $k \equiv 0 \pmod 4$, $k-1 \equiv 3 \pmod 4$, and $k-2 \equiv 2 \pmod 4$, by the inductive hypothesis, this is a game that the first player at that point (who is the second player in our game) can win. **13.** Let $P(n)$ be the statement that exactly $n-1$ moves are required to assemble a puzzle with n pieces. Now $P(1)$ is trivially true. Assume that $P(j)$ is true for all $j \le k$, and consider a puzzle with $k+1$ pieces. The final move must be the joining of two blocks, of size j and $k+1-j$ for some integer j with $1 \le j \le k$. By the inductive hypothesis, it required $j-1$ moves to construct the one block, and $k+1-j-1 = k-j$ moves to construct the other. Therefore, $1+(j-1)+(k-j) = k$ moves are required in all, so $P(k+1)$ is true. **15.** Let the Chomp board have n rows and n columns. We claim that the first player can win the game by making the first move to leave just the top row and leftmost column. Let $P(n)$ be the statement that if a player has presented his opponent with a Chomp configuration consisting of just n cookies in the top row and n cookies in the leftmost column, then he can win the game. We will prove $\forall n\, P(n)$ by strong induction. We know that $P(1)$ is true, because the opponent is forced to take the poisoned cookie at his first turn. Fix $k \ge 1$ and assume that $P(j)$ is true for all $j \le k$. We claim that $P(k+1)$ is true. It is the opponent's turn to move. If she picks the poisoned cookie, then the game is over and she loses. Otherwise, assume she picks the cookie in the top row in column j, or the cookie in the left column in row j, for some j with $2 \le j \le k+1$. The first player now picks the cookie in the left column in row j, or the cookie in the top row in column j, respectively. This leaves the position covered by $P(j-1)$ for his opponent, so by the inductive hypothesis, he can win. **17.** Let $P(n)$ be the statement that if a simple polygon with n sides is triangulated, then at least two of the triangles in the triangulation have two sides that border the exterior of the polygon. We will prove $\forall n \ge 4\, P(n)$. The statement is clearly true for $n = 4$, because there is only one diagonal, leaving two triangles with the desired property. Fix $k \ge 4$ and assume that $P(j)$ is true for all j with $4 \le j \le k$. Consider a polygon with $k+1$ sides, and some triangulation of it. Pick one of the diagonals in this triangulation. First suppose that this diagonal divides the polygon into one triangle and one polygon with k sides. Then the triangle has two sides that border the exterior. Furthermore, the k-gon has, by the inductive hypothesis, two triangles that have two sides that border the exterior of that k-gon, and only one of these triangles can fail to be a triangle that has two sides that border the exterior of the original polygon. The only other case is that this diagonal divides the polygon into two polygons with j sides and $k+3-j$ sides for some j with $4 \le j \le k-1$. By the inductive hypothesis, each of these two polygons has two triangles that have two sides that border their exterior, and in each case only one of these triangles can fail to be a triangle that has two sides that border the exterior of the original polygon. **19.** Let $P(n)$ be the statement that the area of a simple polygon with n sides and vertices all at lattice points is given by $I(P) + B(P)/2 - 1$. We will prove $P(n)$ for all $n \ge 3$. We begin with an additivity lemma: If P is a simple polygon with all vertices at lattice points, divided into polygons P_1 and P_2 by a diagonal, then $I(P) + B(P)/2 - 1 = [I(P_1) + B(P_1)/2 - 1] + [I(P_2) + B(P_2)/2 - 1]$. To prove this, suppose there are k lattice points on the diagonal, not counting its endpoints. Then $I(P) = I(P_1) + I(P_2) + k$ and $B(P) = B(P_1) + B(P_2) - 2k - 2$; and the result follows by simple algebra. What this says in particular is that if Pick's formula gives the correct area for P_1 and P_2, then it must give the correct formula for P, whose area is the sum of the areas for P_1 and P_2; and similarly if Pick's formula gives the correct area for P and one of the P_i's, then it must give the correct formula for the other P_i. Next we prove the theorem for rectangles whose sides are parallel to the coordinate axes. Such a rectangle necessarily has vertices at (a, b), (a, c), (d, b), and (d, c), where a, b, c, and d are integers with $b < c$ and $a < d$. Its area is $(c-b)(d-a)$. Also, $B = 2(c-b+d-a)$ and $I = (c-b-1)(d-a-1) = (c-b)(d-a)-(c-b)-(d-a)+1$. Therefore, $I + B/2 - 1 = (c-b)(d-a) - (c-b) - (d-a) + 1 + (c-b+d-a) - 1 = (c-b)(d-a)$, which is the desired area. Next consider a right triangle whose legs are parallel to the coordinate axes. This triangle is half a rectangle of the type just considered, for which Pick's formula holds, so by the additivity lemma, it holds for the triangle as well. (The values of B and I are the same for each of the two triangles, so if Picks's formula gave an answer that was either too small or too large, then it would give a correspondingly wrong answer for the rectangle.) For the next step, consider an arbitrary triangle with vertices at lattice points that is not of the type already considered. Embed it in as small a rectangle as possible. There are several possible ways this can happen, but in any case (and adding one more edge in one case), the rectangle will have been partitioned into the given triangle and two or three right triangles with sides parallel to the coordinate axes. Again by the additivity lemma, we are guaranteed that Pick's formula gives the correct area for the given triangle. This completes the proof of $P(3)$, the basis step in our strong induction proof. For the inductive step, given an arbitrary polygon, use Lemma 1 in the text to split it into two polygons. Then by the additivity lemma above and the inductive hypothesis, we know that Pick's formula gives the correct area for this polygon. **21. a)** In the left figure $\angle abp$ is smallest, but $\overline{bp}$ is not an interior diagonal. **b)** In the right figure $\overline{bd}$ is not an interior diagonal. **c)** In the right figure $\overline{bd}$ is not an interior diagonal. **23. a)** When we try to prove the inductive step and find a triangle in each subpolygon with at least two sides bordering the exterior, it may happen in each case that the triangle we are guaranteed in fact borders the diagonal (which is part of the boundary of that polygon). This leaves us with no triangles guaranteed to touch the boundary of the *original* polygon. **b)** We proved the stronger statement $\forall n \ge 4\, T(n)$ in Exercise 17. **25. a)** The inductive step here allows us to conclude that

$P(3), P(5), \ldots$ are all true, but we can conclude nothing about $P(2), P(4), \ldots$. **b)** $P(n)$ is true for all positive integers n, using strong induction. **c)** The inductive step here enables us to conclude that $P(2)$, $P(4)$, $P(8)$, $P(16)$, …are all true, but we can conclude nothing about $P(n)$ when n is not a power of 2. **d)** This is mathematical induction; we can conclude that $P(n)$ is true for all positive integers n. **27.** Suppose, for a proof by contradiction, that there is some positive integer n such that $P(n)$ is not true. Let m be the smallest positive integer greater than n for which $P(m)$ *is* true; we know that such an m exists because $P(m)$ is true for infinitely many values of m. But we know that $P(m) \to P(m-1)$, so $P(m-1)$ is also true. Thus, $m-1$ cannot be greater than n, so $m-1=n$ and $P(n)$ is in fact true. This contradiction shows that $P(n)$ is true for all n. **29.** The error is in going from the base case $n=0$ to the next case, $n=1$; we cannot write 1 as the sum of two smaller natural numbers. **31.** Assume that the well-ordering property holds. Suppose that $P(1)$ is true and that the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(n)] \to P(n+1)$ is true for every positive integer n. Let S be the set of positive integers n for which $P(n)$ is false. We will show $S=\emptyset$. Assume that $S \neq \emptyset$. Then by the well-ordering property there is a least integer m in S. We know that m cannot be 1 because $P(1)$ is true. Because $n=m$ is the least integer such that $P(n)$ is false, $P(1), P(2), \ldots, P(m-1)$ are true, and $m-1 \geq 1$. Because $[P(1) \wedge P(2) \wedge \cdots \wedge P(m-1)] \to P(m)$ is true, it follows that $P(m)$ must also be true, which is a contradiction. Hence, $S=\emptyset$. **33.** In each case, give a proof by contradiction based on a "smallest counterexample," that is, values of n and k such that $P(n,k)$ is not true and n and k are smallest in some sense. **a)** Choose a counterexample with $n+k$ as small as possible. We cannot have $n=1$ and $k=1$, because we are given that $P(1,1)$ is true. Therefore, either $n>1$ or $k>1$. In the former case, by our choice of counterexample, we know that $P(n-1,k)$ is true. But the inductive step then forces $P(n,k)$ to be true, a contradiction.The latter case is similar. So our supposition that there is a counterexample mest be wrong, and $P(n,k)$ is true in all cases. **b)** Choose a counterexample with n as small as possible. We cannot have $n=1$, because we are given that $P(1,k)$ is true for all k. Therefore, $n>1$. By our choice of counterexample, we know that $P(n-1,k)$ is true. But the inductive step then forces $P(n,k)$ to be true, a contradiction. **c)** Choose a counterexample with k as small as possible. We cannot have $k=1$, because we are given that $P(n,1)$ is true for all n. Therefore, $k>1$. By our choice of counterexample, we know that $P(n,k-1)$ is true. But the inductive step then forces $P(n,k)$ to be true, a contradiction. **35.** Let $P(n)$ be the statement that if $x_1, x_2, \ldots, x_n$ are n distinct real numbers, then $n-1$ multiplications are used to find the product of these numbers no matter how parentheses are inserted in the product. We will prove that $P(n)$ is true using strong induction. The basis case $P(1)$ is true because $1-1=0$ multiplications are required to find the product of x_1, a product with only one factor. Suppose that $P(k)$ is true for $1 \leq k \leq n$. The last multiplication used to find the product of the $n+1$ distinct real numbers $x_1, x_2, \ldots, x_n, x_{n+1}$ is a multiplication of the product of the first k of these numbers for some k and the product of the last $n+1-k$ of them. By the inductive hypothesis, $k-1$ multiplications are used to find the product of k of the numbers, no matter how parentheses were inserted in the product of these numbers, and $n-k$ multiplications are used to find the product of the other $n+1-k$ of them, no matter how parentheses were inserted in the product of these numbers. Because one more multiplication is required to find the product of all $n+1$ numbers, the total number of multiplications used equals $(k-1)+(n-k)+1=n$. Hence, $P(n+1)$ is true. **37.** Assume that $a=dq+r=dq'+r'$ with $0 \leq r < d$ and $0 \leq r' < d$. Then $d(q-q')=r'-r$. It follows that d divides $r'-r$. Because $-d < r'-r < d$, we have $r'-r=0$. Hence, $r'=r$. It follows that $q=q'$. **39.** This is a paradox caused by self-reference. The answer is clearly "no." There are a finite number of English words, so only a finite number of strings of 15 words or fewer; therefore, only a finite number of positive integers can be so described, not all of them. **41.** Suppose that the well-ordering property were false. Let S be a nonempty set of nonnegative integers that has no least element. Let $P(n)$ be the statement "$i \notin S$ for $i=0,1,\ldots,n$." $P(0)$ is true because if $0 \in S$ then S has a least element, namely, 0. Now suppose that $P(n)$ is true. Thus, $0 \notin S, 1 \notin S, \ldots, n \notin S$. Clearly, $n+1$ cannot be in S, for if it were, it would be its least element. Thus $P(n+1)$ is true. So by the principle of mathematical induction, $n \notin S$ for all nonnegative integers n. Thus, $S=\emptyset$, a contradiction. **43.** Strong induction implies the principle of mathematical induction, for if one has shown that $P(k) \to P(k+1)$ is true, then one has also shown that $[P(1) \wedge \cdots \wedge P(k)] \to P(k+1)$ is true. By Exercise 41, the principle of mathematical induction implies the well-ordering property. Therefore by assuming strong induction as an axiom, we can prove the well-ordering property.

Section 5.3

1. a) $f(1)=3, f(2)=5, f(3)=7, f(4)=9$ **b)** $f(1)=3, f(2)=9, f(3)=27, f(4)=81$ **c)** $f(1)=2, f(2)=4, f(3)=16, f(4)=65{,}536$ **d)** $f(1)=3, f(2)=13, f(3)=183, f(4)=33{,}673$ **3. a)** $f(2)=-1, f(3)=5, f(4)=2, f(5)=17$ **b)** $f(2)=-4, f(3)=32, f(4)=-4096, f(5)=536{,}870{,}912$ **c)** $f(2)=8, f(3)=176, f(4)=92{,}672, f(5)=25{,}764{,}174{,}848$ **d)** $f(2)=-\frac{1}{2}, f(3)=-4, f(4)=\frac{1}{8}, f(5)=-32$ **5. a)** Not valid **b)** $f(n)=1-n$. *Basis step:* $f(0)=1=1-0$. *Inductive step:* if $f(k)=1-k$, then $f(k+1)=f(k)-1=1-k-1=1-(k+1)$. **c)** $f(n)=4-n$ if $n>0$, and $f(0)=2$. *Basis step:* $f(0)=2$ and $f(1)=3=4-1$. *Inductive step* (with $k \geq 1$): $f(k+1)=f(k)-1=(4-k)-1=4-(k+1)$. **d)** $f(n)=2^{\lfloor(n+1)/2\rfloor}$. *Basis step:* $f(0)=1=2^{\lfloor(0+1)/2\rfloor}$ and $f(1)=2=2^{\lfloor(1+1)/2\rfloor}$. *Inductive step* (with $k \geq 1$): $f(k+1)=2f(k-1)=2 \cdot 2^{\lfloor k/2\rfloor}=2^{\lfloor k/2\rfloor+1}=2^{\lfloor((k+1)+1)/2\rfloor}$. **e)** $f(n)=3^n$. *Basis step:* Trivial. *Inductive step:* For odd n, $f(n)=3f(n-1)=3 \cdot 3^{n-1}=3^n$; and for even $n>1$, $f(n)=9f(n-2)=9 \cdot 3^{n-2}=3^n$. **7.** There are many possible correct answers. We will supply relatively simple ones. **a)** $a_{n+1} = a_n + 6$ for $n \geq 1$ and $a_1 = 6$ **b)** $a_{n+1} = a_n + 2$ for $n \geq 1$ and $a_1 = 3$ **c)** $a_{n+1} = 10a_n$ for $n \geq 1$ and $a_1 = 10$ **d)** $a_{n+1} = a_n$ for $n \geq 1$ and $a_1 = 5$ **9.** $F(0) = 0$, $F(n) = F(n-1) + n$ for $n \geq 1$ **11.** $P_m(0) = 0$, $P_m(n+1) = P_m(n) + m$ **13.** Let $P(n)$ be "$f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$." *Basis step:* $P(1)$ is true because $f_1 = 1 = f_2$. *Inductive step:* Assume that $P(k)$ is true. Then $f_1 + f_3 + \cdots + f_{2k-1} + f_{2k+1} = f_{2k} + f_{2k+1} = f_{2k+2} + f_{2(k+1)}$. **15.** *Basis step:* $f_0 f_1 + f_1 f_2 = 0 \cdot 1 + 1 \cdot 1 = 1^2 = f_2^2$. *Inductive step:* Assume that $f_0 f_1 + f_1 f_2 + \cdots + f_{2k-1} f_{2k} = f_{2k}^2$. Then $f_0 f_1 + f_1 f_2 + \cdots + f_{2k-1} f_{2k} + f_{2k} f_{2k+1} + f_{2k+1} f_{2k+2} = f_{2k}^2 + f_{2k} f_{2k+1} + f_{2k+1} f_{2k+2} = f_{2k}(f_{2k} + f_{2k+1}) + f_{2k+1} f_{2k+2} = f_{2k} f_{2k+2} + f_{2k+1} f_{2k+2} = (f_{2k} + f_{2k+1}) f_{2k+2} = f_{2k+2}^2$. **17.** The number of divisions used by the Euclidean algorithm to find $\gcd(f_{n+1}, f_n)$ is 0 for $n = 0$, 1 for $n = 1$, and $n - 1$ for $n \geq 2$. To prove this result for $n \geq 2$ we use mathematical induction. For $n = 2$, one division shows that $\gcd(f_3, f_2) = \gcd(2, 1) = \gcd(1, 0) = 1$. Now assume that $k - 1$ divisions are used to find $\gcd(f_{k+1}, f_k)$. To find $\gcd(f_{k+2}, f_{k+1})$, first divide f_{k+2} by f_{k+1} to obtain $f_{k+2} = 1 \cdot f_{k+1} + f_k$. After one div- ision we have $\gcd(f_{k+2}, f_{k+1}) = \gcd(f_{k+1}, f_k)$. By the inductive hypothesis it follows that exactly $k - 1$ more divisions are required. This shows that k divisions are required to find $\gcd(f_{k+2}, f_{k+1})$, finishing the inductive proof. **19.** $|A| = -1$. Hence, $|A^n| = (-1)^n$. It follows that $f_{n+1} f_{n-1} - f_n^2 = (-1)^n$. **21. a)** Proof by induction. *Basis step:* For $n = 1$, $\max(-a_1) = -a_1 = -\min(a_1)$. For $n = 2$, there are two cases. If $a_2 \geq a_1$, then $-a_1 \geq -a_2$, so $\max(-a_1, -a_2) = -a_1 = -\min(a_1, a_2)$. If $a_2 < a_1$, then $-a_1 < -a_2$, so $\max(-a_1, -a_2) = -a_2 = -\min(a_1, a_2)$. *Inductive step:* Assume true for k with $k \geq 2$. Then $\max(-a_1, -a_2, \ldots, -a_k, -a_{k+1}) = \max(\max(-a_1, \ldots, -a_k), -a_{k+1}) = \max(-\min(a_1, \ldots, a_k), -a_{k+1}) = -\min(\min(a_1, \ldots, a_k), a_{k+1}) = -\min(a_1, \ldots, a_{k+1})$. **b)** Proof by mathematical induction. *Basis step:* For $n = 1$, the result is the identity $a_1 + b_1 = a_1 + b_1$. For $n = 2$, first consider the case in which $a_1 + b_1 \geq a_2 + b_2$. Then $\max(a_1 + b_1, a_2 + b_2) = a_1 + b_1$. Also note that $a_1 \leq \max(a_1, a_2)$ and $b_1 \leq \max(b_1, b_2)$, so $a_1 + b_1 \leq \max(a_1, a_2) + \max(b_1, b_2)$. Therefore, $\max(a_1+b_1, a_2+b_2) = a_1+b_1 \leq \max(a_1, a_2)+\max(b_1, b_2)$. The case with $a_1 + b_1 < a_2 + b_2$ is similar. *Inductive step:* Assume that the result is true for k. Then $\max(a_1 + b_1, a_2 + b_2, \ldots, a_k + b_k, a_{k+1} + b_{k+1}) = \max(\max(a_1 + b_1, a_2 + b_2, \ldots, a_k + b_k), a_{k+1} + b_{k+1}) \leq \max(\max(a_1, a_2, \ldots, a_k) + \max(b_1, b_2, \ldots, b_k), a_{k+1} + b_{k+1}) \leq \max(\max(a_1, a_2, \ldots, a_k), a_{k+1}) + \max(\max(b_1, b_2, \ldots, b_k), b_{k+1}) = \max(a_1, a_2, \ldots, a_k, a_{k+1}) + \max(b_1, b_2, \ldots, b_k, b_{k+1})$. **c)** Same as part (b), but replace every occurrence of "max" by "min" and invert each inequality. **23.** $5 \in S$, and $x + y \in S$ if $x, y \in S$. **25. a)** $0 \in S$, and if $x \in S$, then $x + 2 \in S$ and $x - 2 \in S$. **b)** $2 \in S$, and if $x \in S$, then $x + 3 \in S$. **c)** $1 \in S$, $2 \in S$, $3 \in S$, $4 \in S$, and if $x \in S$, then $x + 5 \in S$. **27. a)** (0, 1), (1, 1), (2, 1); (0, 2), (1, 2), (2, 2), (3, 2), (4, 2); (0, 3), (1, 3), (2, 3), (3, 3), (4, 3), (5, 3), (6, 3); (0, 4), (1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 4), (7, 4), (8, 4) **b)** Let $P(n)$ be the statement that $a \leq 2b$ whenever $(a, b) \in S$ is obtained by n applications of the recursive step. *Basis step:* $P(0)$ is true, because the only element of S obtained with no applications of the recursive step is (0, 0), and indeed $0 \leq 2 \cdot 0$. *Inductive step:* Assume that $a \leq 2b$ whenever $(a, b) \in S$ is obtained by k or fewer applications of the recursive step, and consider an element obtained with $k + 1$ applications of the recursive step. Because the final application of the recursive step to an element (a, b) must be applied to an element obtained with fewer applications of the recursive step, we know that $a \leq 2b$. Add $0 \leq 2$, $1 \leq 2$, and $2 \leq 2$, respectively, to obtain $a \leq 2(b+1)$, $a + 1 \leq 2(b+1)$, and $a + 2 \leq 2(b+1)$, as desired. **c)** This holds for the basis step, because $0 \leq 0$. If this holds for (a, b), then it also holds for the elements obtained from (a, b) in the recursive step, because adding $0 \leq 2$, $1 \leq 2$, and $2 \leq 2$, respectively, to $a \leq 2b$ yields $a \leq 2(b+1)$, $a + 1 \leq 2(b+1)$, and $a + 2 \leq 2(b+1)$. **29. a)** Define S by $(1, 1) \in S$, and if $(a, b) \in S$, then $(a+2, b) \in S$, $(a, b+2) \in S$, and $(a+1, b+1) \in S$. All elements put in S satisfy the condition, because (1, 1) has an even sum of coordinates, and if (a, b) has an even sum of coordinates, then so do $(a+2, b)$, $(a, b+2)$, and $(a+1, b+1)$. Conversely, we show by induction on the sum of the coordinates that if $a + b$ is even, then $(a, b) \in S$. If the sum is 2, then $(a, b) = (1, 1)$, and the basis step put (a, b) into S. Otherwise the sum is at least 4, and at least one of $(a-2, b)$, $(a, b-2)$, and $(a-1, b-1)$ must have positive integer coordinates whose sum is an even number smaller than $a + b$, and therefore must be in S. Then one application of the recursive step shows that $(a, b) \in S$. **b)** Define S by (1, 1), (1, 2), and (2, 1) are in S, and if $(a, b) \in S$, then $(a+2, b)$ and $(a, b+2)$ are in S. To prove that our definition works, we note first that (1, 1), (1, 2), and (2, 1) all have an odd coordinate, and if (a, b) has an odd coordinate, then so do $(a+2, b)$ and $(a, b+2)$. Conversely, we show by induction on the sum of the coordinates that if (a, b) has at least one odd coordinate, then $(a, b) \in S$. If $(a, b) = (1, 1)$ or $(a, b) = (1, 2)$ or $(a, b) = (2, 1)$, then the basis step put (a, b) into S. Otherwise either a or b is at least 3, so at least one of $(a-2, b)$ and $(a, b-2)$ must have positive integer coordinates whose sum is smaller than $a + b$, and therefore must be in S. Then one application of the recursive step shows that $(a, b) \in S$. **c)** $(1, 6) \in S$ and $(2, 3) \in S$, and if $(a, b) \in S$, then $(a+2, b) \in S$ and $(a, b+6) \in S$. To prove that our definition works, we note first that (1, 6) and (2, 3) satisfy the condition, and if (a, b) satisfies the condition, then so do $(a+2, b)$ and $(a, b+6)$. Conversely we show by induction on the sum of the coordinates that if (a, b) satisfies the condition, then $(a, b) \in S$. For sums 5 and 7, the only points are (1, 6), which the basis step put into S, (2, 3), which the basis step put into S, and $(4, 3) = (2+2, 3)$, which is in S by one application of the recursive definition. For a sum greater than 7, either $a \geq 3$, or

$a \le 2$ and $b \ge 9$, in which case either $(a-2, b)$ or $(a, b-6)$ must have positive integer coordinates whose sum is smaller than $a+b$ and satisfy the condition for being in S. Then one application of the recursive step shows that $(a, b) \in S$. **31.** If x is a set or a variable representing a set, then x is a well-formed formula. If x and y are well-formed formulae, then so are $\overline{x}$, $(x \cup y)$, $(x \cap y)$, and $(x - y)$. **33. a)** If $x \in D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, then $m(x) = x$; if $s = tx$, where $t \in D^*$ and $x \in D$, then $m(s) = \min(m(s), x)$. **b)** Let $t = wx$, where $w \in D^*$ and $x \in D$. If $w = \lambda$, then $m(st) = m(sx) = \min(m(s), x) = \min(m(s), m(x))$ by the recursive step and the basis step of the definition of m. Otherwise, $m(st) = m((sw)x) = \min(m(sw), x)$ by the definition of m. Now $m(sw) = \min(m(s), m(w))$ by the inductive hypothesis of the structural induction, so $m(st) = \min(\min(m(s), m(w)), x) = \min(m(s), \min(m(w), x))$ by the meaning of min. But $\min(m(w), x) = m(wx) = m(t)$ by the recursive step of the definition of m. Thus, $m(st) = \min(m(s), m(t))$. **35.** $\lambda^R = \lambda$ and $(ux)^R = xu^R$ for $x \in \Sigma$, $u \in \Sigma^*$. **37.** $w^0 = \lambda$ and $w^{n+1} = ww^n$. **39.** When the string consists of n 0s followed by n 1s for some non- negative integer n **41.** Let $P(i)$ be "$l(w^i) = i \cdot l(w)$." $P(0)$ is true because $l(w^0) = 0 = 0 \cdot l(w)$. Assume $P(i)$ is true. Then $l(w^{i+1}) = l(ww^i) = l(w) + l(w^i) = l(w) + i \cdot l(w) = (i+1) \cdot l(w)$. **43.** *Basis step:* For the full binary tree consisting of just a root the result is true because $n(T) = 1$ and $h(T) = 0$, and $1 \ge 2 \cdot 0 + 1$. *Inductive step:* Assume that $n(T_1) \ge 2h(T_1) + 1$ and $n(T_2) \ge 2h(T_2) + 1$. By the recursive definitions of $n(T)$ and $h(T)$, we have $n(T) = 1 + n(T_1) + n(T_2)$ and $h(T) = 1 + \max(h(T_1), h(T_2))$. Therefore $n(T) = 1 + n(T_1) + n(T_2) \ge 1 + 2h(T_1) + 1 + 2h(T_2) + 1 \ge 1 + 2 \cdot \max(h(T_1), h(T_2)) + 2 = 1 + 2(\max(h(T_1), h(T_2)) + 1) = 1 + 2h(T)$. **45.** *Basis step:* $a_{0,0} = 0 = 0 + 0$. *Inductive step:* Assume that $a_{m',n'} = m' + n'$ whenever (m', n') is less than (m, n) in the lexicographic ordering of $\mathbf{N} \times \mathbf{N}$. If $n = 0$ then $a_{m,n} = a_{m-1,n} + 1 = m - 1 + n + 1 = m + n$. If $n > 0$, then $a_{m,n} = a_{m,n-1} + 1 = m + n - 1 + 1 = m + n$. **47. a)** $P_{m,m} = P_m$ because a number exceeding m cannot be used in a partition of m. **b)** Because there is only one way to partition 1, namely, $1 = 1$, it follows that $P_{1,n} = 1$. Because there is only one way to partition m into 1s, $P_{m,1} = 1$. When $n > m$ it follows that $P_{m,n} = P_{m,m}$ because a number exceeding m cannot be used. $P_{m,m} = 1 + P_{m,m-1}$ because one extra partition, namely, $m = m$, arises when m is allowed in the partition. $P_{m,n} = P_{m,n-1} + P_{m-n,n}$ if $m > n$ because a partition of m into integers not exceeding n either does not use any ns and hence, is counted in $P_{m,n-1}$ or else uses an n and a partition of $m-n$, and hence, is counted in $P_{m-n,n}$. **c)** $P_5 = 7$, $P_6 = 11$ **49.** Let $P(n)$ be "$A(n, 2) = 4$." *Basis step:* $P(1)$ is true because $A(1, 2) = A(0, A(1, 1)) = A(0, 2) = 2 \cdot 2 = 4$. *Inductive step:* Assume that $P(n)$ is true, that is, $A(n, 2) = 4$. Then $A(n+1, 2) = A(n, A(n+1, 1)) = A(n, 2) = 4$. **51. a)** 16 **b)** 65,536 **53.** Use a double induction argument to prove the stronger statement: $A(m, k) > A(m, l)$ when $k > l$. *Basis step:* When $m = 0$ the statement is true because $k > l$ implies that $A(0, k) = 2k > 2l = A(0, l)$. *Inductive step:* Assume that $A(m, x) > A(m, y)$ for all nonnegative integers x and y with $x > y$. We will show that this implies that $A(m+1, k) > A(m+1, l)$ if $k > l$. *Basis steps:* When $l = 0$ and $k > 0$, $A(m+1, l) = 0$ and either $A(m+1, k) = 2$ or $A(m+1, k) = A(m, A(m+1, k-1))$. If $m = 0$, this is $2A(1, k-1) = 2^k$. If $m > 0$, this is greater than 0 by the inductive hypothesis. In all cases, $A(m+1, k) > 0$, and in fact, $A(m+1, k) \ge 2$. If $l = 1$ and $k > 1$, then $A(m+1, l) = 2$ and $A(m+1, k) = A(m, A(m+1, k-1))$, with $A(m+1, k-1) \ge 2$. Hence, by the inductive hypothesis, $A(m, A(m+1, k-1)) \ge A(m, 2) > A(m, 1) = 2$. *Inductive step:* Assume that $A(m+1, r) > A(m+1, s)$ for all $r > s$, $s = 0, 1, \ldots, l$. Then if $k+1 > l+1$ it follows that $A(m+1, k+1) = A(m, A(m+1, k)) > A(m, A(m+1, k)) = A(m+1, l+1)$. **55.** From Exercise 54 it follows that $A(i, j) \ge A(i-1, j) \ge \cdots \ge A(0, j) = 2j \ge j$. **57.** Let $P(n)$ be "$F(n)$ is well-defined." Then $P(0)$ is true because $F(0)$ is specified. Assume that $P(k)$ is true for all $k < n$. Then $F(n)$ is well-defined at n because $F(n)$ is given in terms of $F(0), F(1), \ldots, F(n-1)$. So $P(n)$ is true for all integers n. **59. a)** The value of $F(1)$ is ambiguous. **b)** $F(2)$ is not defined because $F(0)$ is not defined. **c)** $F(3)$ is ambiguous and $F(4)$ is not defined because $F(\frac{4}{3})$ makes no sense. **d)** The definition of $F(1)$ is ambiguous because both the second and third clause seem to apply. **e)** $F(2)$ cannot be computed because trying to compute $F(2)$ gives $F(2) = 1 + F(F(1)) = 1 + F(2)$. **61. a)** 1 **b)** 2 **c)** 3 **d)** 3 **e)** 4 **f)** 4 **g)** 5 **63.** $f_0^*(n) = \lceil n/a \rceil$ **65.** $f_2^*(n) = \lceil \log \log n \rceil$ for $n \ge 2$, $f_2^*(1) = 0$

Section 5.4

1. First, we use the recursive step to write $5! = 5 \cdot 4!$. We then use the recursive step repeatedly to write $4! = 4 \cdot 3!$, $3! = 3 \cdot 2!$, $2! = 2 \cdot 1!$, and $1! = 1 \cdot 0!$. Inserting the value of $0! = 1$, and working back through the steps, we see that $1! = 1 \cdot 1 = 1$, $2! = 2 \cdot 1! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2! = 3 \cdot 2 = 6$, $4! = 4 \cdot 3! = 4 \cdot 6 = 24$, and $5! = 5 \cdot 4! = 5 \cdot 24 = 120$. **3.** With this input, the algorithm uses the **else** clause to find that $\gcd(8, 13) = \gcd(13 \textbf{ mod } 8, 8) = \gcd(5, 8)$. It uses this clause again to find that $\gcd(5, 8) = \gcd(8 \textbf{ mod } 5, 5) = \gcd(3, 5)$, then to get $\gcd(3, 5) = \gcd(5 \textbf{ mod } 3, 3) = \gcd(2, 3)$, then $\gcd(2, 3) = \gcd(3 \textbf{ mod } 2, 2) = \gcd(1, 2)$, and once more to get $\gcd(1, 2) = \gcd(2 \textbf{ mod } 1, 1) = \gcd(0, 1)$. Finally, to find $\gcd(0, 1)$ it uses the first step with $a = 0$ to find that $\gcd(0, 1) = 1$. Consequently, the algorithm finds that $\gcd(8, 13) = 1$. **5.** First, because $n = 11$ is odd, we use the **else** clause to see that $mpower(3, 11, 5) = (mpower(3, 5, 5)^2 \textbf{ mod } 5 \cdot 3 \textbf{ mod } 5) \textbf{ mod } 5$. We next use the **else** clause again to see that $mpower(3, 5, 5) = (mpower(3, 2, 5)^2 \textbf{ mod } 5 \cdot 3 \textbf{ mod } 5) \textbf{ mod } 5$. Then we use the **else if** clause to see that $mpower(3, 2, 5) = mpower(3, 1, 5)^2 \textbf{ mod } 5$. Using the **else** clause again, we have $mpower(3, 1, 5) = (mpower(3, 0, 5)^2 \textbf{ mod } 5 \cdot 3 \textbf{ mod } 5) \textbf{ mod } 5$. Finally, us-