Exercises

1. Use a direct proof to show that the sum of two odd integers is even.
2. Use a direct proof to show that the sum of two even integers is even.
3. Show that the square of an even number is an even number using a direct proof.
4. Show that the additive inverse, or negative, of an even number is an even number using a direct proof.
5. Prove that if $m + n$ and $n + p$ are even integers, where m, n, and p are integers, then $m + p$ is even. What kind of proof did you use?
6. Use a direct proof to show that the product of two odd numbers is odd.
7. Use a direct proof to show that every odd integer is the difference of two squares.
8. Prove that if n is a perfect square, then $n + 2$ is not a perfect square.
9. Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.
10. Use a direct proof to show that the product of two rational numbers is rational.
11. Prove or disprove that the product of two irrational numbers is irrational.
12. Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.
13. Prove that if x is irrational, then $1/x$ is irrational.
14. Prove that if x is rational and $x \neq 0$, then $1/x$ is rational.
15. Use a proof by contraposition to show that if $x + y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$.
16. Prove that if m and n are integers and mn is even, then m is even or n is even.
17. Show that if n is an integer and $n^3 + 5$ is odd, then n is even using
    - **a)** a proof by contraposition.
    - **b)** a proof by contradiction.
18. Prove that if n is an integer and $3n + 2$ is even, then n is even using
    - **a)** a proof by contraposition.
    - **b)** a proof by contradiction.
19. Prove the proposition $P(0)$, where $P(n)$ is the proposition "If n is a positive integer greater than 1, then $n^2 > n$." What kind of proof did you use?
20. Prove the proposition $P(1)$, where $P(n)$ is the proposition "If n is a positive integer, then $n^2 \geq n$." What kind of proof did you use?
21. Let $P(n)$ be the proposition "If a and b are positive real numbers, then $(a + b)^n \geq a^n + b^n$." Prove that $P(1)$ is true. What kind of proof did you use?
22. Show that if you pick three socks from a drawer containing just blue socks and black socks, you must get either a pair of blue socks or a pair of black socks.
23. Show that at least ten of any 64 days chosen must fall on the same day of the week.
24. Show that at least three of any 25 days chosen must fall in the same month of the year.
25. Use a proof by contradiction to show that there is no rational number r for which $r^3 + r + 1 = 0$. [*Hint:* Assume that $r = a/b$ is a root, where a and b are integers and a/b is in lowest terms. Obtain an equation involving integers by multiplying by b^3. Then look at whether a and b are each odd or even.]
26. Prove that if n is a positive integer, then n is even if and only if $7n + 4$ is even.
27. Prove that if n is a positive integer, then n is odd if and only if $5n + 6$ is odd.
28. Prove that $m^2 = n^2$ if and only if $m = n$ or $m = -n$.
29. Prove or disprove that if m and n are integers such that $mn = 1$, then either $m = 1$ and $n = 1$, or else $m = -1$ and $n = -1$.
30. Show that these three statements are equivalent, where a and b are real numbers: (*i*) a is less than b, (*ii*) the average of a and b is greater than a, and (*iii*) the average of a and b is less than b.
31. Show that these statements about the integer x are equivalent: (*i*) $3x + 2$ is even, (*ii*) $x + 5$ is odd, (*iii*) x^2 is even.
32. Show that these statements about the real number x are equivalent: (*i*) x is rational, (*ii*) $x/2$ is rational, (*iii*) $3x - 1$ is rational.
33. Show that these statements about the real number x are equivalent: (*i*) x is irrational, (*ii*) $3x + 2$ is irrational, (*iii*) $x/2$ is irrational.
34. Is this reasoning for finding the solutions of the equation $\sqrt{2x^2 - 1} = x$ correct? (*1*) $\sqrt{2x^2 - 1} = x$ is given; (*2*) $2x^2 - 1 = x^2$, obtained by squaring both sides of (1); (*3*) $x^2 - 1 = 0$, obtained by subtracting x^2 from both sides of (2); (*4*) $(x - 1)(x + 1) = 0$, obtained by factoring the left-hand side of $x^2 - 1$; (*5*) $x = 1$ or $x = -1$, which follows because $ab = 0$ implies that $a = 0$ or $b = 0$.
35. Are these steps for finding the solutions of $\sqrt{x + 3} = 3 - x$ correct? (*1*) $\sqrt{x + 3} = 3 - x$ is given; (*2*) $x + 3 = x^2 - 6x + 9$, obtained by squaring both sides of (1); (*3*) $0 = x^2 - 7x + 6$, obtained by subtracting $x + 3$ from both sides of (2); (*4*) $0 = (x - 1)(x - 6)$, obtained by factoring the right-hand side of (3); (*5*) $x = 1$ or $x = 6$, which follows from (4) because $ab = 0$ implies that $a = 0$ or $b = 0$.
36. Show that the propositions p_1, p_2, p_3, and p_4 can be shown to be equivalent by showing that $p_1 \leftrightarrow p_4$, $p_2 \leftrightarrow p_3$, and $p_1 \leftrightarrow p_3$.
37. Show that the propositions p_1, p_2, p_3, p_4, and p_5 can be shown to be equivalent by proving that the conditional statements $p_1 \rightarrow p_4$, $p_3 \rightarrow p_1$, $p_4 \rightarrow p_2$, $p_2 \rightarrow p_5$, and $p_5 \rightarrow p_3$ are true.

Exercises

1. Prove that $n^2 + 1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$.
2. Prove that there are no positive perfect cubes less than 1000 that are the sum of the cubes of two positive integers.
3. Prove that if x and y are real numbers, then $\max(x, y) + \min(x, y) = x + y$. [*Hint:* Use a proof by cases, with the two cases corresponding to $x \geq y$ and $x < y$, respectively.]
4. Use a proof by cases to show that $\min(a, \min(b, c)) = \min(\min(a, b), c)$ whenever a, b, and c are real numbers.
5. Prove using the notion of without loss of generality that $\min(x, y) = (x + y - |x - y|)/2$ and $\max(x, y) = (x + y + |x - y|)/2$ whenever x and y are real numbers.
6. Prove using the notion of without loss of generality that $5x + 5y$ is an odd integer when x and y are integers of opposite parity.
7. Prove the **triangle inequality**, which states that if x and y are real numbers, then $|x| + |y| \geq |x + y|$ (where $|x|$ represents the absolute value of x, which equals x if $x \geq 0$ and equals $-x$ if $x < 0$).
8. Prove that there is a positive integer that equals the sum of the positive integers not exceeding it. Is your proof constructive or nonconstructive?
9. Prove that there are 100 consecutive positive integers that are not perfect squares. Is your proof constructive or nonconstructive?
10. Prove that either $2 \cdot 10^{500} + 15$ or $2 \cdot 10^{500} + 16$ is not a perfect square. Is your proof constructive or nonconstructive?
11. Prove that there exists a pair of consecutive integers such that one of these integers is a perfect square and the other is a perfect cube.
12. Show that the product of two of the numbers $65^{1000} - 8^{2001} + 3^{177}$, $79^{1212} - 9^{2399} + 2^{2001}$, and $24^{4493} - 5^{8192} + 7^{1777}$ is nonnegative. Is your proof constructive or nonconstructive? [*Hint:* Do not try to evaluate these numbers!]
13. Prove or disprove that there is a rational number x and an irrational number y such that x^y is irrational.
14. Prove or disprove that if a and b are rational numbers, then a^b is also rational.
15. Show that each of these statements can be used to express the fact that there is a unique element x such that $P(x)$ is true. [Note that we can also write this statement as $\exists! x P(x)$.]
    a) $\exists x \forall y (P(y) \leftrightarrow x = y)$
    b) $\exists x P(x) \wedge \forall x \forall y (P(x) \wedge P(y) \rightarrow x = y)$
    c) $\exists x (P(x) \wedge \forall y (P(y) \rightarrow x = y))$
16. Show that if a, b, and c are real numbers and $a \neq 0$, then there is a unique solution of the equation $ax + b = c$.
17. Suppose that a and b are odd integers with $a \neq b$. Show there is a unique integer c such that $|a - c| = |b - c|$.
18. Show that if r is an irrational number, there is a unique integer n such that the distance between r and n is less than $1/2$.
19. Show that if n is an odd integer, then there is a unique integer k such that n is the sum of $k - 2$ and $k + 3$.
20. Prove that given a real number x there exist unique numbers n and ϵ such that $x = n + \epsilon$, n is an integer, and $0 \leq \epsilon < 1$.
21. Prove that given a real number x there exist unique numbers n and ϵ such that $x = n - \epsilon$, n is an integer, and $0 \leq \epsilon < 1$.
22. Use forward reasoning to show that if x is a nonzero real number, then $x^2 + 1/x^2 \geq 2$. [*Hint:* Start with the inequality $(x - 1/x)^2 \geq 0$ which holds for all nonzero real numbers x.]
23. The **harmonic mean** of two real numbers x and y equals $2xy/(x + y)$. By computing the harmonic and geometric means of different pairs of positive real numbers, formulate a conjecture about their relative sizes and prove your conjecture.
24. The **quadratic mean** of two real numbers x and y equals $\sqrt{(x^2 + y^2)/2}$. By computing the arithmetic and quadratic means of different pairs of positive real numbers, formulate a conjecture about their relative sizes and prove your conjecture.

*25. Write the numbers $1, 2, \ldots, 2n$ on a blackboard, where n is an odd integer. Pick any two of the numbers, j and k, write $|j - k|$ on the board and erase j and k. Continue this process until only one integer is written on the board. Prove that this integer must be odd.

*26. Suppose that five ones and four zeros are arranged around a circle. Between any two equal bits you insert a 0 and between any two unequal bits you insert a 1 to produce nine new bits. Then you erase the nine original bits. Show that when you iterate this procedure, you can never get nine zeros. [*Hint:* Work backward, assuming that you did end up with nine zeros.]

27. Formulate a conjecture about the decimal digits that appear as the final decimal digit of the fourth power of an integer. Prove your conjecture using a proof by cases.
28. Formulate a conjecture about the final two decimal digits of the square of an integer. Prove your conjecture using a proof by cases.
29. Prove that there is no positive integer n such that $n^2 + n^3 = 100$.
30. Prove that there are no solutions in integers x and y to the equation $2x^2 + 5y^2 = 14$.
31. Prove that there are no solutions in positive integers x and y to the equation $x^4 + y^4 = 625$.
32. Prove that there are infinitely many solutions in positive integers x, y, and z to the equation $x^2 + y^2 = z^2$. [*Hint:* Let $x = m^2 - n^2$, $y = 2mn$, and $z = m^2 + n^2$, where m and n are integers.]

33. Adapt the proof in Example 4 in Section 1.7 to prove that if $n = abc$, where a, b, and c are positive integers, then $a \leq \sqrt[3]{n}$, $b \leq \sqrt[3]{n}$, or $c \leq \sqrt[3]{n}$.

34. Prove that $\sqrt[3]{2}$ is irrational.

35. Prove that between every two rational numbers there is an irrational number.

36. Prove that between every rational number and every irrational number there is an irrational number.

***37.** Let $S = x_1y_1 + x_2y_2 + \cdots + x_ny_n$, where $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are orderings of two different sequences of positive real numbers, each containing n elements.

- **a)** Show that S takes its maximum value over all orderings of the two sequences when both sequences are sorted (so that the elements in each sequence are in nondecreasing order).
- **b)** Show that S takes its minimum value over all orderings of the two sequences when one sequence is sorted into nondecreasing order and the other is sorted into nonincreasing order.

38. Prove or disprove that if you have an 8-gallon jug of water and two empty jugs with capacities of 5 gallons and 3 gallons, respectively, then you can measure 4 gallons by successively pouring some of or all of the water in a jug into another jug.

39. Verify the $3x + 1$ conjecture for these integers.

a) 6 **b)** 7 **c)** 17 **d)** 21

40. Verify the $3x + 1$ conjecture for these integers.

a) 16 **b)** 11 **c)** 35 **d)** 113

41. Prove or disprove that you can use dominoes to tile the standard checkerboard with two adjacent corners removed (that is, corners that are not opposite).

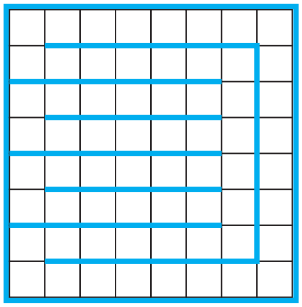
42. Prove or disprove that you can use dominoes to tile a standard checkerboard with all four corners removed.

43. Prove that you can use dominoes to tile a rectangular checkerboard with an even number of squares.

44. Prove or disprove that you can use dominoes to tile a 5×5 checkerboard with three corners removed.

45. Use a proof by exhaustion to show that a tiling using dominoes of a 4×4 checkerboard with opposite corners removed does not exist. [*Hint:* First show that you can assume that the squares in the upper left and lower right corners are removed. Number the squares of the original checkerboard from 1 to 16, starting in the first row, moving right in this row, then starting in the leftmost square in the second row and moving right, and so on. Remove squares 1 and 16. To begin the proof, note that square 2 is covered either by a domino laid horizontally, which covers squares 2 and 3, or vertically, which covers squares 2 and 6. Consider each of these cases separately, and work through all the subcases that arise.]

***46.** Prove that when a white square and a black square are removed from an 8×8 checkerboard (colored as in the text) you can tile the remaining squares of the checkerboard using dominoes. [*Hint:* Show that when one black and one white square are removed, each part of the partition of the remaining cells formed by inserting the barriers shown in the figure can be covered by dominoes.]



47. Show that by removing two white squares and two black squares from an 8×8 checkerboard (colored as in the text) you can make it impossible to tile the remaining squares using dominoes.

***48.** Find all squares, if they exist, on an 8×8 checkerboard such that the board obtained by removing one of these square can be tiled using straight triominoes. [*Hint:* First use arguments based on coloring and rotations to eliminate as many squares as possible from consideration.]

***49.** **a)** Draw each of the five different tetrominoes, where a tetromino is a polyomino consisting of four squares.

- **b)** For each of the five different tetrominoes, prove or disprove that you can tile a standard checkerboard using these tetrominoes.

***50.** Prove or disprove that you can tile a 10×10 checkerboard using straight tetrominoes.

Key Terms and Results

TERMS

proposition: a statement that is true or false

propositional variable: a variable that represents a proposition

truth value: true or false

$\neg p$ (negation of p): the proposition with truth value opposite to the truth value of p

logical operators: operators used to combine propositions

compound proposition: a proposition constructed by combining propositions using logical operators

truth table: a table displaying all possible truth values of propositions

$p \vee q$ (disjunction of p and q): the proposition "p or q," which is true if and only if at least one of p and q is true