$H(x)$ true, but we cannot conclude that Lola is one such x. **19. a)** Fallacy of affirming the conclusion **b)** Fallacy of begging the question **c)** Valid argument using modus tollens **d)** Fallacy of denying the hypothesis **21.** By the second premise, there is some lion that does not drink coffee. Let Leo be such a creature. By simplification we know that Leo is a lion. By modus ponens we know from the first premise that Leo is fierce. Hence, Leo is fierce and does not drink coffee. By the definition of the existential quantifier, there exist fierce creatures that do not drink coffee, that is, some fierce creatures do not drink coffee. **23.** The error occurs in step (5), because we cannot assume, as is being done here, that the c that makes P true is the same as the c that makes Q true. **25.** We are given the premises $\forall x(P(x) \to Q(x))$ and $\neg Q(a)$. We want to show $\neg P(a)$. Suppose, to the contrary, that $\neg P(a)$ is not true. Then $P(a)$ is true. Therefore by universal modus ponens, we have $Q(a)$. But this contradicts the given premise $\neg Q(a)$. Therefore our supposition must have been wrong, and so $\neg P(a)$ is true, as desired.

27.

Step	Reason
1. $\forall x(P(x) \wedge R(x))$	Premise
2. $P(a) \wedge R(a)$	Universal instantiation from (1)
3. $P(a)$	Simplification from (2)
4. $\forall x(P(x) \to (Q(x) \wedge S(x)))$	Premise
5. $Q(a) \wedge S(a)$	Universal modus ponens from (3) and (4)
6. $S(a)$	Simplification from (5)
7. $R(a)$	Simplification from (2)
8. $R(a) \wedge S(a)$	Conjunction from (7) and (6)
9. $\forall x(R(x) \wedge S(x))$	Universal generalization from (5)

29.

Step	Reason
1. $\exists x \neg P(x)$	Premise
2. $\neg P(c)$	Existential instantiation from (1)
3. $\forall x(P(x) \vee Q(x))$	Premise
4. $P(c) \vee Q(c)$	Universal instantiation from (3)
5. $Q(c)$	Disjunctive syllogism from (4) and (2)
6. $\forall x(\neg Q(x) \vee S(x))$	Premise
7. $\neg Q(c) \vee S(c)$	Universal instantiation from (6)
8. $S(c)$	Disjunctive syllogism from (5) and (7)
9. $\forall x(R(x) \to \neg S(x))$	Premise
10. $R(c) \to \neg S(c)$	Universal instantiation from (9)
11. $\neg R(c)$	Modus tollens from (8) and (10)
12. $\exists x \neg R(x)$	Existential generalization from (11)

31. Let p be "It is raining"; let q be "Yvette has her umbrella"; let r be "Yvette gets wet." Assumptions are $\neg p \vee q$, $\neg q \vee \neg r$, and $p \vee \neg r$. Resolution on the first two gives $\neg p \vee \neg r$. Resolution on this and the third assumption gives $\neg r$, as desired. **33.** Assume that this proposition is satisfiable. Using resolution on the first two clauses enables us to conclude $q \vee q$; in other words, we know that q has to be true. Using resolution on the last two clauses enables us to conclude $\neg q \vee \neg q$; in other words, we know that $\neg q$ has to be true. This is a contradiction. So this proposition is not satisfiable. **35.** Valid

Section 1.7

1. Let $n = 2k + 1$ and $m = 2l + 1$ be odd integers. Then $n+m = 2(k+l+1)$ is even. **3.** Suppose that n is even. Then $n = 2k$ for some integer k. Therefore, $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. Because we have written n^2 as 2 times an integer, we conclude that n^2 is even. **5.** Direct proof: Suppose that $m+n$ and $n+p$ are even. Then $m+n = 2s$ for some integer s and $n + p = 2t$ for some integer t. If we add these, we get $m + p + 2n = 2s + 2t$. Subtracting $2n$ from both sides and factoring, we have $m + p = 2s + 2t - 2n = 2(s + t - n)$. Because we have written $m + p$ as 2 times an integer, we conclude that $m + p$ is even. **7.** Because n is odd, we can write $n = 2k + 1$ for some integer k. Then $(k+1)^2 - k^2 = k^2 + 2k + 1 - k^2 = 2k + 1 = n$. **9.** Suppose that r is rational and i is irrational and $s = r + i$ is rational. Then by Example 7, $s + (-r) = i$ is rational, which is a contradiction. **11.** Because $\sqrt{2} \cdot \sqrt{2} = 2$ is rational and $\sqrt{2}$ is irrational, the product of two irrational numbers is not necessarily irrational. **13.** Proof by contraposition: If $1/x$ were rational, then by definition $1/x = p/q$ for some integers p and q with $q \neq 0$. Because $1/x$ cannot be 0 (if it were, then we'd have the contradiction $1 = x \cdot 0$ by multiplying both sides by x), we know that $p \neq 0$. Now $x = 1/(1/x) = 1/(p/q) = q/p$ by the usual rules of algebra and arithmetic. Hence, x can be written as the quotient of two integers with the denominator nonzero. Thus by definition, x is rational. **15.** Assume that it is not true that $x \geq 1$ or $y \geq 1$. Then $x < 1$ and $y < 1$. Adding these two inequalities, we obtain $x + y < 2$, which is the negation of $x + y \geq 2$. **17. a)** Assume that n is odd, so $n = 2k+1$ for some integer k. Then $n^3 + 5 = 2(4k^3 + 6k^2 + 3k + 3)$. Because $n^3 + 5$ is two times some integer, it is even. **b)** Suppose that $n^3 + 5$ is odd and n is odd. Because n is odd and the product of two odd numbers is odd, it follows that n^2 is odd and then that n^3 is odd. But then $5 = (n^3 + 5) - n^3$ would have to be even because it is the difference of two odd numbers. Therefore, the supposition that $n^3 + 5$ and n were both odd is wrong. **19.** The proposition is vacuously true because 0 is not a positive integer. Vacuous proof. **21.** $P(1)$ is true because $(a + b)^1 = a + b \geq a^1 + b^1 = a + b$. Direct proof. **23.** If we chose 9 or fewer days on each day of the week, this would account for at most $9 \cdot 7 = 63$ days. But we chose 64 days. This contradiction shows that at least 10 of the days we chose must be on the same day of the week. **25.** Suppose by way of contradiction that a/b is a rational root, where a and b are integers and this fraction is in lowest terms (that is, a and b have no common divisor greater than 1). Plug this proposed root into the equation to obtain $a^3/b^3 + a/b + 1 = 0$. Multiply through by b^3 to obtain $a^3 + ab^2 + b^3 = 0$. If a and b are both odd, then the left-hand side is the sum of three odd numbers and therefore must be odd. If a is odd and b is even, then the left-hand side is odd + even + even, which is again odd. Similarly, if a is even and b is odd, then the left-hand side is even + even + odd, which is again odd. Because the fraction a/b is in simplest terms, it cannot happen that both a and b are even. Thus in all cases, the left-hand side is odd, and therefore cannot equal 0. This contradiction shows that no such root exists. **27.** First, assume that n is odd, so that $n = 2k+1$ for some integer k. Then $5n+6 = 5(2k+1)+6 = 10k + 11 = 2(5k + 5) + 1$. Hence, $5n + 6$ is odd. To prove the converse, suppose that n is even, so that $n = 2k$ for some integer k. Then $5n + 6 = 10k + 6 = 2(5k + 3)$, so $5n + 6$ is even. Hence, n is odd if and only if $5n + 6$ is odd. **29.** This proposition is true. Suppose that m is neither 1 nor -1. Then mn has a factor m larger than 1. On the other hand, $mn = 1$, and 1 has no such factor. Hence, $m = 1$ or $m = -1$. In the first case $n = 1$, and in the second case $n = -1$, because $n = 1/m$. **31.** We prove that all these are equivalent to x being even. If x is even, then $x = 2k$ for some integer k. Therefore $3x+2 = 3\cdot 2k+2 = 6k+2 = 2(3k+1)$, which is even, because it has been written in the form $2t$, where $t = 3k+1$. Similarly, $x + 5 = 2k + 5 = 2k + 4 + 1 = 2(k + 2) + 1$, so $x + 5$ is odd; and $x^2 = (2k)^2 = 2(2k^2)$, so x^2 is even. For the converses, we will use a proof by contraposition. So assume that x is not even; thus x is odd and we can write $x = 2k + 1$ for some integer k. Then $3x+2 = 3(2k+1)+2 = 6k+5 = 2(3k+2)+1$, which is odd (i.e., not even), because it has been written in the form $2t+1$, where $t = 3k+2$. Similarly, $x+5 = 2k+1+5 = 2(k+3)$, so $x + 5$ is even (i.e., not odd). That x^2 is odd was already proved in Example 1. **33.** We give proofs by contraposition of $(i) \to (ii)$, $(ii) \to (i)$, $(i) \to (iii)$, and $(iii) \to (i)$. For the first of these, suppose that $3x + 2$ is rational, namely, equal to p/q for some integers p and q with $q \neq 0$. Then we can write $x = ((p/q) - 2)/3 = (p - 2q)/(3q)$, where $3q \neq 0$. This shows that x is rational. For the second conditional statement, suppose that x is rational, namely, equal to p/q for some integers p and q with $q \neq 0$. Then we can write $3x+2 = (3p+2q)/q$, where $q \neq 0$. This shows that $3x+2$ is rational. For the third conditional statement, suppose that $x/2$ is rational, namely, equal to p/q for some integers p and q with $q \neq 0$. Then we can write $x = 2p/q$, where $q \neq 0$. This shows that x is rational. And for the fourth conditional statement, suppose that x is rational, namely, equal to p/q for some integers p and q with $q \neq 0$. Then we can write $x/2 = p/(2q)$, where $2q \neq 0$. This shows that $x/2$ is rational. **35.** No **37.** Suppose that $p_1 \to p_4 \to p_2 \to p_5 \to p_3 \to p_1$. To prove that one of these propositions implies any of the others, just use hypothetical syllogism repeatedly. **39.** We will give a proof by contradiction. Suppose that $a_1, a_2, \ldots, a_n$ are all less than A, where A is the average of these numbers. Then $a_1 + a_2 + \cdots + a_n < nA$. Dividing both sides by n shows that $A = (a_1 + a_2 + \cdots + a_n)/n < A$, which is a contradiction. **41.** We will show that the four statements are equivalent by showing that (i) implies (ii), (ii) implies (iii), (iii) implies (iv), and (iv) implies (i). First, assume that n is even. Then $n = 2k$ for some integer k. Then $n + 1 = 2k + 1$, so $n + 1$ is odd. This shows that (i) implies (ii). Next, suppose that $n + 1$ is odd, so $n + 1 = 2k + 1$ for some integer k. Then $3n + 1 = 2n + (n + 1) = 2(n + k) + 1$, which shows that $3n+1$ is odd, showing that (ii) implies (iii). Next, suppose that $3n + 1$ is odd, so $3n + 1 = 2k + 1$ for some integer k. Then $3n = (2k + 1) - 1 = 2k$, so $3n$ is even. This shows that (iii) implies (iv). Finally, suppose that n is not even. Then n is odd, so $n = 2k + 1$ for some integer k. Then $3n = 3(2k+1) = 6k+3 = 2(3k+1)+1$, so $3n$ is odd. This completes a proof by contraposition that (iv) implies (i).

Section 1.8

1. $1^2 + 1 = 2 \geq 2 = 2^1$; $2^2 + 1 = 5 \geq 4 = 2^2$; $3^2 + 1 = 10 \geq 8 = 2^3$; $4^2 + 1 = 17 \geq 16 = 2^4$ **3.** If $x \leq y$, then $\max(x, y) + \min(x, y) = y + x = x + y$. If $x \geq y$, then $\max(x, y) + \min(x, y) = x + y$. Because these are the only two cases, the equality always holds. **5.** Because $|x - y| = |y - x|$, the values of x and y are interchangeable. Therefore, without loss of generality, we can assume that $x \geq y$. Then $(x + y - (x - y))/2 = (x + y - x + y)/2 = 2y/2 = y = \min(x, y)$. Similarly, $(x + y + (x - y))/2 = (x + y + x - y)/2 = 2x/2 = x = \max(x, y)$. **7.** There are four cases. *Case 1:* $x \geq 0$ and $y \geq 0$. Then $|x| + |y| = x + y = |x + y|$. *Case 2:* $x < 0$ and $y < 0$. Then $|x| + |y| = -x + (-y) = -(x + y) = |x + y|$ because $x+y < 0$. *Case 3:* $x \geq 0$ and $y < 0$. Then $|x|+|y| = x+(-y)$. If $x \geq -y$, then $|x + y| = x + y$. But because $y < 0$, $-y > y$, so $|x| + |y| = x + (-y) > x + y = |x + y|$. If $x < -y$, then $|x+y| = -(x+y) = -x+(-y)$. But because $x \geq 0$, $x \geq -x$, so $|x|+|y| = x + (-y) \geq -x + (-y) = |x+y|$. *Case 4:* $x < 0$ and $y \geq 0$. Identical to Case 3 with the roles of x and y reversed. **9.** 10,001, 10,002, ..., 10,100 are all nonsquares, because $100^2 = 10{,}000$ and $101^2 = 10{,}201$; constructive. **11.** $8 = 2^3$ and $9 = 3^2$ **13.** Let $x = 2$ and $y = \sqrt{2}$. If $x^y = 2^{\sqrt{2}}$ is irrational, we are done. If not, then let $x = 2^{\sqrt{2}}$ and $y = \sqrt{2}/4$. Then $x^y = (2^{\sqrt{2}})^{\sqrt{2}/4} = 2^{\sqrt{2}\cdot(\sqrt{2})/4} = 2^{1/2} = \sqrt{2}$. **15. a)** This statement asserts the existence of x with a certain property. If we let $y = x$, then we see that $P(x)$ is true. If y is anything other than x, then $P(x)$ is not true. Thus, x is the unique element that makes P true. **b)** The first clause here says that there is an element that makes P true. The second clause says that whenever two elements both make P true, they are in fact the same element. Together these say that P is satisfied by exactly one element. **c)** This statement asserts the existence of an x that makes P true and has the further property that whenever we find an element that makes P true, that element is x. In other words, x is the unique element that makes P true. **17.** The equation $|a - c| = |b - c|$ is equivalent to the disjunction of two equations: $a - c = b - c$ or $a - c = -b + c$. The first of these is equivalent to $a = b$, which contradicts the assumptions made in this problem, so the original equation is equivalent to $a - c = -b + c$. By adding $b + c$ to both sides and dividing by 2, we see that this equation is equivalent to $c = (a + b)/2$. Thus, there is a unique solution. Furthermore, this c is an integer, because the sum of the odd integers a and b is even. **19.** We are being asked to solve $n = (k-2) + (k+3)$ for k. Using the usual, reversible, rules of algebra, we see that this equation is equivalent to $k = (n-1)/2$. In other words, this is the one and only value of k that makes our equation true. Because n is odd, $n-1$ is even, so k is an integer. **21.** If x is itself an integer, then we can take $n = x$ and $\epsilon = 0$. No other solution is possible in this case, because if the integer n is greater than x, then n is at least $x+1$, which would make $\epsilon \geq 1$. If x is not an integer, then round it up to the next integer, and call that integer n. Let $\epsilon = n - x$. Clearly $0 \leq \epsilon < 1$; this is the only ϵ that will work with this n, and n cannot be any larger, because ϵ is constrained to be less than 1. **23.** The harmonic mean of distinct positive real numbers x and y is always less than their geometric mean. To prove $2xy/(x+y) < \sqrt{xy}$, multiply both sides by $(x+y)/(2\sqrt{xy})$ to obtain the equivalent inequality $\sqrt{xy} < (x+y)/2$, which is proved in Example 14. **25.** The parity (oddness or evenness) of the sum of the numbers written on the board never changes, because $j+k$ and $|j-k|$ have the same parity (and at each step we reduce the sum by $j+k$ but increase it by $|j-k|$). Therefore the integer at the end of the process must have the same parity as $1 + 2 + \cdots + (2n) = n(2n+1)$, which is odd because n is odd. **27.** Without loss of generality we can assume that n is nonnegative, because the fourth power of an integer and the fourth power of its negative are the same. We divide an arbitrary positive integer n by 10, obtaining a quotient k and remainder l, whence $n = 10k + l$, and l is an integer between 0 and 9, inclusive. Then we compute n^4 in each of these 10 cases. We get the following values, where X is some integer that is a multiple of 10, whose exact value we do not care about. $(10k+0)^4 = 10{,}000k^4 = 10{,}000k^4 + 0$, $(10k+1)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 1$, $(10k+2)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 16$, $(10k+3)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 81$, $(10k+4)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 256$, $(10k+5)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 625$, $(10k+6)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 1296$, $(10k+7)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 2401$, $(10k+8)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 4096$, $(10k+9)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 6561$. Because each coefficient indicated by X is a multiple of 10, the corresponding term has no effect on the ones digit of the answer. Therefore the ones digits are 0, 1, 6, 1, 6, 5, 6, 1, 6, 1, respectively, so it is always a 0, 1, 5, or 6. **29.** Because $n^3 > 100$ for all $n > 4$, we need only note that $n = 1$, $n = 2$, $n = 3$, and $n = 4$ do not satisfy $n^2 + n^3 = 100$. **31.** Because $5^4 = 625$, both x and y must be less than 5. Then $x^4 + y^4 \leq 4^4 + 4^4 = 512 < 625$. **33.** If it is not true that $a \leq \sqrt[3]{n}$, $b \leq \sqrt[3]{n}$, or $c \leq \sqrt[3]{n}$, then $a > \sqrt[3]{n}$, $b > \sqrt[3]{n}$, and $c > \sqrt[3]{n}$. Multiplying these inequalities of positive numbers together we obtain $abc < (\sqrt[3]{n})^3 = n$, which implies the negation of our hypothesis that $n = abc$. **35.** By finding a common denominator, we can assume that the given rational numbers are a/b and c/b, where b is a positive integer and a and c are integers with $a < c$. In particular, $(a+1)/b \leq c/b$. Thus, $x = (a + \frac{1}{2}\sqrt{2})/b$ is between the two given rational numbers, because $0 < \sqrt{2} < 2$. Furthermore, x is irrational, because if x were rational, then $2(bx - a) = \sqrt{2}$ would be as well, in violation of Example 10 in Section 1.7. **37. a)** Without loss of generality, we can assume that the x sequence is already sorted into nondecreasing order, because we can relabel the indices. There are only a finite number of possible orderings for the y sequence, so if we can show that we can increase the sum (or at least keep it the same) whenever we find y_i and y_j that are out of order (i.e., $i < j$ but $y_i > y_j$) by switching them, then we will have shown that the sum is largest when the y sequence is in nondecreasing order. Indeed, if we perform the swap, then we have added $x_i y_j + x_j y_i$ to the sum and subtracted $x_i y_i + x_j y_j$. The net effect is to have added $x_i y_j + x_j y_i - x_i y_i - x_j y_j = (x_j - x_i)(y_i - y_j)$, which is nonnegative by our ordering assumptions. **b)** Similar to part (a) **39. a)** $6 \to 3 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$ **b)** $7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$ **c)** $17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$ **d)** $21 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$ **41.** Without loss of generality, assume that the upper left and upper right corners of the board are removed. Place three dominoes horizontally to fill the remaining portion of the first row, and fill each of the other seven rows with four horizontal dominoes. **43.** Because there is an even number of squares in all, either there is an even number of squares in each row or there is an even number of squares in each column. In the former case, tile the board in the obvious way by placing the dominoes horizontally, and in the latter case, tile the board in the obvious way by placing the dominoes vertically. **45.** We can rotate the board if necessary to make the removed squares be 1 and 16. Square 2 must be covered by a domino. If that domino is placed to cover squares 2 and 6, then the following domino placements are forced in succession: 5-9, 13-14, and 10-11, at which point there is no way to cover square 15. Otherwise, square 2 must be covered by a domino placed at 2-3. Then the following domino placements are forced: 4-8, 11-12, 6-7, 5-9, and 10-14, and again there is no way to cover square 15. **47.** Remove the two black squares adjacent to a white corner, and remove two white squares other than that corner. Then no domino can cover that white corner.

49. a)