Exercises

1. Let A be the set of students who live within one mile of school and let B be the set of students who walk to classes. Describe the students in each of these sets.

a) $A \cap B$ **b)** $A \cup B$
c) $A - B$ **d)** $B - A$

2. Suppose that A is the set of sophomores at your school and B is the set of students in discrete mathematics at your school. Express each of these sets in terms of A and B.

a) the set of sophomores taking discrete mathematics in your school
b) the set of sophomores at your school who are not taking discrete mathematics
c) the set of students at your school who either are sophomores or are taking discrete mathematics
d) the set of students at your school who either are not sophomores or are not taking discrete mathematics

3. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find

a) $A \cup B$. **b)** $A \cap B$.
c) $A - B$. **d)** $B - A$.

4. Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Find

a) $A \cup B$. **b)** $A \cap B$.
c) $A - B$. **d)** $B - A$.

In Exercises 5–10 assume that A is a subset of some underlying universal set U.

5. Prove the complementation law in Table 1 by showing that $\overline{\overline{A}} = A$.

6. Prove the identity laws in Table 1 by showing that

a) $A \cup \emptyset = A$. **b)** $A \cap U = A$.

7. Prove the domination laws in Table 1 by showing that

a) $A \cup U = U$. **b)** $A \cap \emptyset = \emptyset$.

8. Prove the idempotent laws in Table 1 by showing that

a) $A \cup A = A$. **b)** $A \cap A = A$.

9. Prove the complement laws in Table 1 by showing that

a) $A \cup \overline{A} = U$. **b)** $A \cap \overline{A} = \emptyset$.

10. Show that

a) $A - \emptyset = A$. **b)** $\emptyset - A = \emptyset$.

11. Let A and B be sets. Prove the commutative laws from Table 1 by showing that

a) $A \cup B = B \cup A$.
b) $A \cap B = B \cap A$.

12. Prove the first absorption law from Table 1 by showing that if A and B are sets, then $A \cup (A \cap B) = A$.

13. Prove the second absorption law from Table 1 by showing that if A and B are sets, then $A \cap (A \cup B) = A$.

14. Find the sets A and B if $A - B = \{1, 5, 7, 8\}$, $B - A = \{2, 10\}$, and $A \cap B = \{3, 6, 9\}$.

15. Prove the second De Morgan law in Table 1 by showing that if A and B are sets, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$

a) by showing each side is a subset of the other side.
b) using a membership table.

16. Let A and B be sets. Show that

a) $(A \cap B) \subseteq A$. **b)** $A \subseteq (A \cup B)$.
c) $A - B \subseteq A$. **d)** $A \cap (B - A) = \emptyset$.
e) $A \cup (B - A) = A \cup B$.

17. Show that if A, B, and C are sets, then $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$

a) by showing each side is a subset of the other side.
b) using a membership table.

18. Let A, B, and C be sets. Show that

a) $(A \cup B) \subseteq (A \cup B \cup C)$.
b) $(A \cap B \cap C) \subseteq (A \cap B)$.
c) $(A - B) - C \subseteq A - C$.
d) $(A - C) \cap (C - B) = \emptyset$.
e) $(B - A) \cup (C - A) = (B \cup C) - A$.

19. Show that if A and B are sets, then

a) $A - B = A \cap \overline{B}$.
b) $(A \cap B) \cup (A \cap \overline{B}) = A$.

20. Show that if A and B are sets with $A \subseteq B$, then

a) $A \cup B = B$.
b) $A \cap B = A$.

21. Prove the first associative law from Table 1 by showing that if A, B, and C are sets, then $A \cup (B \cup C) = (A \cup B) \cup C$.

22. Prove the second associative law from Table 1 by showing that if A, B, and C are sets, then $A \cap (B \cap C) = (A \cap B) \cap C$.

23. Prove the first distributive law from Table 1 by showing that if A, B, and C are sets, then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

24. Let A, B, and C be sets. Show that $(A - B) - C = (A - C) - (B - C)$.

25. Let $A = \{0, 2, 4, 6, 8, 10\}$, $B = \{0, 1, 2, 3, 4, 5, 6\}$, and $C = \{4, 5, 6, 7, 8, 9, 10\}$. Find

a) $A \cap B \cap C$. **b)** $A \cup B \cup C$.
c) $(A \cup B) \cap C$. **d)** $(A \cap B) \cup C$.

26. Draw the Venn diagrams for each of these combinations of the sets A, B, and C.

a) $A \cap (B \cup C)$ **b)** $\overline{A} \cap \overline{B} \cap \overline{C}$
c) $(A - B) \cup (A - C) \cup (B - C)$

27. Draw the Venn diagrams for each of these combinations of the sets A, B, and C.

a) $A \cap (B - C)$ **b)** $(A \cap B) \cup (A \cap C)$
c) $(A \cap \overline{B}) \cup (A \cap \overline{C})$

28. Draw the Venn diagrams for each of these combinations of the sets A, B, C, and D.

a) $(A \cap B) \cup (C \cap D)$ **b)** $\overline{A} \cup \overline{B} \cup \overline{C} \cup \overline{D}$
c) $A - (B \cap C \cap D)$

29. What can you say about the sets A and B if we know that

a) $A \cup B = A$? **b)** $A \cap B = A$?
c) $A - B = A$? **d)** $A \cap B = B \cap A$?
e) $A - B = B - A$?

30. Can you conclude that $A = B$ if A, B, and C are sets such that

a) $A \cup C = B \cup C$? **b)** $A \cap C = B \cap C$?
c) $A \cup C = B \cup C$ and $A \cap C = B \cap C$?

31. Let A and B be subsets of a universal set U. Show that $A \subseteq B$ if and only if $\overline{B} \subseteq \overline{A}$.

The **symmetric difference** of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.

32. Find the symmetric difference of $\{1, 3, 5\}$ and $\{1, 2, 3\}$.

33. Find the symmetric difference of the set of computer science majors at a school and the set of mathematics majors at this school.

34. Draw a Venn diagram for the symmetric difference of the sets A and B.

35. Show that $A \oplus B = (A \cup B) - (A \cap B)$.

36. Show that $A \oplus B = (A - B) \cup (B - A)$.

37. Show that if A is a subset of a universal set U, then

a) $A \oplus A = \emptyset$. **b)** $A \oplus \emptyset = A$.
c) $A \oplus U = \overline{A}$. **d)** $A \oplus \overline{A} = U$.

38. Show that if A and B are sets, then

a) $A \oplus B = B \oplus A$. **b)** $(A \oplus B) \oplus B = A$.

39. What can you say about the sets A and B if $A \oplus B = A$?

***40.** Determine whether the symmetric difference is associative; that is, if A, B, and C are sets, does it follow that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$?

***41.** Suppose that A, B, and C are sets such that $A \oplus C = B \oplus C$. Must it be the case that $A = B$?

42. If A, B, C, and D are sets, does it follow that $(A \oplus B) \oplus (C \oplus D) = (A \oplus C) \oplus (B \oplus D)$?

43. If A, B, C, and D are sets, does it follow that $(A \oplus B) \oplus (C \oplus D) = (A \oplus D) \oplus (B \oplus C)$?

44. Show that if A and B are finite sets, then $A \cup B$ is a finite set.

45. Show that if A is an infinite set, then whenever B is a set, $A \cup B$ is also an infinite set.

***46.** Show that if A, B, and C are sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

(This is a special case of the inclusion–exclusion principle, which will be studied in Chapter 8.)

47. Let $A_i = \{1, 2, 3, \ldots, i\}$ for $i = 1, 2, 3, \ldots$. Find

a) $\bigcup_{i=1}^{n} A_i$. **b)** $\bigcap_{i=1}^{n} A_i$.

48. Let $A_i = \{\ldots, -2, -1, 0, 1, \ldots, i\}$. Find

a) $\bigcup_{i=1}^{n} A_i$. **b)** $\bigcap_{i=1}^{n} A_i$.

49. Let A_i be the set of all nonempty bit strings (that is, bit strings of length at least one) of length not exceeding i. Find

a) $\bigcup_{i=1}^{n} A_i$. **b)** $\bigcap_{i=1}^{n} A_i$.

50. Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ if for every positive integer i,

a) $A_i = \{i, i + 1, i + 2, \ldots\}$.
b) $A_i = \{0, i\}$.
c) $A_i = (0, i)$, that is, the set of real numbers x with $0 < x < i$.
d) $A_i = (i, \infty)$, that is, the set of real numbers x with $x > i$.

51. Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ if for every positive integer i,

a) $A_i = \{-i, -i + 1, \ldots, -1, 0, 1, \ldots, i - 1, i\}$.
b) $A_i = \{-i, i\}$.
c) $A_i = [-i, i]$, that is, the set of real numbers x with $-i \leq x \leq i$.
d) $A_i = [i, \infty)$, that is, the set of real numbers x with $x \geq i$.

52. Suppose that the universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Express each of these sets with bit strings where the ith bit in the string is 1 if i is in the set and 0 otherwise.

a) $\{3, 4, 5\}$
b) $\{1, 3, 6, 10\}$
c) $\{2, 3, 4, 7, 8, 9\}$

53. Using the same universal set as in the last problem, find the set specified by each of these bit strings.

a) 11 1100 1111
b) 01 0111 1000
c) 10 0000 0001

54. What subsets of a finite universal set do these bit strings represent?

a) the string with all zeros
b) the string with all ones

55. What is the bit string corresponding to the difference of two sets?

56. What is the bit string corresponding to the symmetric difference of two sets?

57. Show how bitwise operations on bit strings can be used to find these combinations of $A = \{a, b, c, d, e\}$, $B = \{b, c, d, g, p, t, v\}$, $C = \{c, e, i, o, u, x, y, z\}$, and $D = \{d, e, h, i, n, o, t, u, x, y\}$.

a) $A \cup B$ **b)** $A \cap B$
c) $(A \cup D) \cap (B \cup C)$ **d)** $A \cup B \cup C \cup D$

58. How can the union and intersection of n sets that all are subsets of the universal set U be found using bit strings?

The **successor** of the set A is the set $A \cup \{A\}$.

59. Find the successors of the following sets.

a) $\{1, 2, 3\}$ **b)** $\emptyset$
c) $\{\emptyset\}$ **d)** $\{\emptyset, \{\emptyset\}\}$