Homework 9 (optional): Solutions

ECS 20 (Winter 2019)

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Exercise 1

Show that $\forall n \in \mathbb{N}, \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Let $P(n)$ be the proposition: $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. Let us also define $LHS(n) = \sum_{i=1}^{n} i^3$ and $RHS(n) = \left(\frac{n(n+1)}{2}\right)^2$

- *Basis step:* $P(1)$ is true:

$$\begin{aligned} LHS(1) &= \sum_{i=1}^{1} i^3 = 1 \\ RHS(1) &= \left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = 1 \end{aligned}$$

- *Inductive step:* Let k be a positive integer ($k \leq 0$), and let us suppose that $P(k)$ is true. We want to show that $P(k+1)$ is true.

  Let us compute $LHS(k+1) = \sum_{i=1}^{k+1} i^3$:

$$
\begin{aligned}
LHS(k+1) &= \sum_{i=1}^{k} i^3 + (k+1)^3 \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\
&= \frac{k^2}{4}(k+1)^2 + (k+1)(k+1)^2 \\
&= \frac{k^2+4k+4}{4}(k+1)^2 \\
&= \frac{(k+2)^2}{4}(k+1)^2 \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2
\end{aligned}
$$

And:

$$RHS(k+1) = \left(\frac{(k+1)(k+2)}{2}\right)^2$$

Therefore $LHS(k+1) = RHS(k+1)$, which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all n.

Exercise 2

Show that $\forall n \in \mathbb{N}, \sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}$.

Let $P(n)$ be the proposition: $\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}$. We define $LHS(n) = \sum_{i=1}^{n} i(i+1)(i+2)$ and $RHS(n) = \frac{n(n+1)(n+2)(n+3)}{4}$

- *Basis step:* $P(1)$ is true:

$$
\begin{aligned}
LHS(1) &= 1 * (1+1) * (1+2) = 6 \\
RHS(1) &= \frac{1 * (1+1) * (1+2) * (1+3)}{4} = 6
\end{aligned}
$$

- *Inductive step:* Let k be a positive integer ($k \leq 0$), and let us suppose that $P(k)$ is true. We want to show that $P(k+1)$ is true.

Let us compute $LHS(k+1)$:

$$\begin{aligned}
LHS(k+1) &= = \sum_{i=1}^{k+1} i(i+1)(i+2) \\
&= LHS(k) + (k+1)(k+2)(k+3) \\
&= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) \\
&= \frac{k(k+1)(k+2)(k+3)}{4} + \frac{4(k+1)(k+2)(k+3)}{4} \\
&= \frac{(k+1)(k+2)(k+3)(k+4)}{4}
\end{aligned}$$

Let us compute $RHS(k+1)$:

$$RHS(k+1) = \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

Therefore $LHS(k+1) = RHS(k+1)$, which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all n.

Exercise 3

Show that $\forall n \in \mathbb{N}, n > 1, \sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}$.

Let $P(n)$ be the proposition: $\sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}$. Let us define $LHS(n) = \sum_{i=1}^{n} \frac{1}{i^2}$ and $RHS(n) = 2 - \frac{1}{n}$. We want to show that $P(n)$ is true for all $n > 1$.

- *Basis step:* We show that $P(2)$ is true:

$$LHS(2) = 1 + \frac{1}{4} = \frac{5}{4}$$
$$RHS(2) = 2 - \frac{1}{2} = \frac{6}{4}$$

  Therefore $LHS(2) < RHS(2)$ and $P(2)$ is true.

- *Inductive step:* Let k be a positive integer greater than 1 ($k > 1$), and let us suppose that $P(k)$ is true. We want to show that $P(k+1)$ is true.

$$LHS(k+1) = LHS(k) + \frac{1}{(k+1)^2}$$

Since P(k) is true, we find:

$$LHS(k+1) < 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

Since $k+1 > k$, $\frac{1}{(k+1)^2} < \frac{1}{k(k+1)}$.
Therefore

$$LHS(k+1) < 2 - \frac{1}{k} + \frac{1}{k(k+1)}$$

We can use the property : $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$:

$$\begin{aligned} LHS(k+1) &< 2 - \frac{1}{k} + \frac{1}{k} - \frac{1}{k+1} \\ LHS(k+1) &< 2 - \frac{1}{k+1} \end{aligned}$$

Since $RHS(k+1) = 2 - \frac{1}{k+1}$, we get $LHS(k+1) < RHS(k+1)$ which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n > 1$.

Exercise 4

Show that $\forall n \in \mathbb{N}, n > 3, n^2 - 7n + 12 \geq 0$.

Let $P(n)$ be the proposition: $n^2 - 7n + 12 \geq 0$. We want to show that $P(n)$ is true for n greater than 3. Let us define $LHS(n) = n^2 - 7n + 12$.
Notice that $LHS(1) = 6$, $LHS(2) = 2$ and $LHS(3) = 0$ hence $P(1)$, $P(2)$ and $P(3)$ are true.

- *Basis step:* $P(4)$ is true:

  $$LHS(4) = 4^2 - 7 * 4 + 12 = 0$$

  Therefore $LHS(4) \geq 0$ and $P(4)$ is true.

- *Inductive step:* Let k be a positive integer greater than 3 ($k > 3$), and let us suppose that $P(k)$ is true. We want to show that $P(k+1)$ is true.

  $$\begin{aligned} LHS(k+1) &= (k+1)^2 - 7(k+1) + 12 \\ &= k^2 + 2k + 1 - 7k - 7 + 12 \\ &= (k^2 - 7k + 12) + (2k - 6) \end{aligned}$$

  Since $P(k)$ is true, we know that $k^2 - 7k + 12 \geq 0$. Since $k \geq 4$, $2k - 6 > 0$. Therefore, $(k+1)^2 - 7(k+1) + 12 > 0$.
  This validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n > 3$.

Exercise 5

Show that $\forall n \in \mathbb{N}, n > 1$, a set S_n with n elements has $\dfrac{n(n-1)}{2}$ subsets that contain exactly two elements.

Let $P(n)$ be the proposition: A set S_n with n elements has $\dfrac{n(n-1)}{2}$ subsets that contain exactly two elements.
We want to show that $P(n)$ is true for all $n \geq 2$; we use a proof by induction.

- *Basis step:* $P(2)$ is true: As the set S_2 contains 2 elements, there is only one subset that containing exactly two elements, and $n(n-1)/2 = 1$.

- *Inductive step:* Let k be a positive integer greater or equal to 2 ($k \geq 2$), and let us suppose that $P(k)$ is true. We want to show that $P(k+1)$ is true.
  Let us consider a set S_{k+1} of $k+1$ elements: $S_{k+1} = \{a_1, a_2, \ldots, a_k, a_{k+1}\}$. Let S_k be the set with the first k elements of S_{k+1}: $S_k = \{a_1, \ldots, a_k\}$. Since $P(k)$ is true, there are $k(k-1)/2$ subsets of S_k that contain exactly two elements.
  The $(k+1)$th element of S_{k+1} a_{k+1} can pair with each of the elements of S_k to build a subset of S_{k+1} of exactly two elements. These new subsets do not duplicate with any of the $k(k-1)/2$ subsets of S_k because the $(k+1)$th element does not appear in any of these subsets. There are no other two-element subsets.
  Therefore, the total number of two-element subsets of S_{k+1} is: $k(k-1)/2 + k = (k(k-1) + 2k)/2 = k(k+1)/2 = (k+1)((k+1)-1)/2$. This validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n \geq 2$.

Exercise 6

Find the flaw with the following proof that : $P(n) : a^n = 1$ for all non negative integer n, whenever a is a non zero real number:

- *Basis step:* $P(0)$ is true: $a^0 = 1$ is true, by definition of a^0

- *Strong Inductive step:* assume that $a^j = 1$ for all non negative integers j with $j \leq k$. Then note that:

$$a^{k+1} = \frac{a^k a^k}{a^{k-1}} = \frac{1 \times 1}{1} = 1$$

  Therefore $P(k+1)$ is true.

The principle of proof by strong mathematical induction allows us to conclude that $P(n)$ is true for all $n \geq 0$.

This is again a case in which if we are not careful, we can prove nearly every thing! In the proof given:

- the basis step is correct: by definition we indeed have $a^0 = 1$.
- Inductive step: the assumption should really be written:
  assume that $a^j = 1$, for all integers j with $0 \leq j \leq k$. When we write $a^{k+1} = \frac{a^k a^k}{a^{k-1}}$, we need to use the premise for $j = k$ and $j = k - 1$. But for $k = 0$, $k - 1 < 0$, and we are outside the limit of validity. This means that we can show $P(k) \to P(k+1)$ only for $k > 0$. This is not enough to apply the method of proof by induction!

Exercise 7

Show that $\forall n \in \mathbb{N}$, 21 divides $4^{n+1} + 5^{2n-1}$.

Let $P(n)$ be the proposition: 21 divides $4^{n+1} + 5^{2n-1}$. We want to show that $P(n)$ is true for all n; we use a proof by induction.

- *Basis step:* $P(1)$ is true: when $n = 1$, $4^{n+1} + 5^{2n-1} = 16 + 5 = 21$ is divisible by 21.
- *Inductive step:* Let k be a positive integer, and let us suppose that $P(k)$ is true. We want to show that $P(k+1)$ is true.

$$\begin{aligned} 4^{(k+1)+1} + 5^{2(k+1)-1} &= 4 * 4^{k+1} + 5^2 * 5^{2k-1} \\ &= 4 * 4^{k+1} + 25 * 5^{2k-1} \\ &= 4(4^{k+1} + 5^{2k-1}) + 21 * 5^{2k-1} \end{aligned}$$

  Because $4^{k+1} + 5^{2k-1}$ and $21 * 5^{2k-1}$ both are divisible by 21, $4^{(k+1)+1} + 5^{2(k+1)-1}$ is also divisible by 21: $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n \geq 0$.

Exercise 8

Show that $\forall n \in \mathbb{N} f_1^2 + f_2^2 + \ldots + f_n^2 = f_n f_{n+1}$ where f_n are the Fibonacci numbers.

Let $P(n)$ be the proposition: $f_1^2 + f_2^2 + \ldots + f_n^2 = f_n f_{n+1}$
where f_n are the Fibonacci numbers. Let us define $LHS(n) = f_1^2 + f_2^2 + \ldots + f_n^2$ and $RHS(n) = f_n f_{n+1}$.

We want to show that $P(n)$ is true for all n; we use a proof by induction.

- *Basis step:* $P(1)$ is true:

$$\begin{aligned} LHS(2) &= f_1^2 = 1^2 = 1 \\ RHS(2) &= f_1 f_2 = 1. \end{aligned}$$

- *Inductive step:* Let k be a positive integer, and let us suppose that $P(k)$ is true. We want to show that $P(k+1)$ is true.
  Then

$$\begin{aligned} LHS(k+1) &= f_1^2 + f_2^2 + ... + f_k^2 + f_{k+1}^2 \\ &= f_k f_{k+1} + f_{k+1}^2 \\ &= f_{k+1}(f_k + f_{k+1}) \\ &= f_{k+1} f_{k+2} \end{aligned}$$

  and

$$RHS(k+1) = f_{k+1} f_{k+2}$$

  Therefore $LHS(k+1) = RHS(k+1)$, which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all n.

Exercise 9

Show that $\forall n \in \mathbb{N} f_0 - f_1 + f_2 - \ldots - f_{2n-1} + f_{2n} = f_{2n-1} - 1$ where f_n are the Fibonacci numbers.

Let $P(n)$ be the proposition: $f_0 - f_1 + f_2 - \ldots - f_{2n-1} + f_{2n} = f_{2n-1} - 1$ where f_n are the Fibonacci numbers. Let us define $LHS(n) = f_0 - f_1 + f_2 - \ldots - f_{2n-1} + f_{2n}$ and $RHS(n) = f_{2n-1} - 1$.

We want to show that $P(n)$ is true for all $n > 0$; we use a proof by induction.

- *Basis step:*

$$LHS(1) = f_0 - f_1 + f_2 = 0 - 1 + 1 = 0$$
$$RHS(1) = f_1 - 1 = 1 - 1 = 0$$

  Therefore $LHS(1) = RHS(1)$ and $P(1)$ is true.

- *Inductive step:* Let k be a positive integer, and let us suppose that $P(k)$ is true. We want to show that $P(k+1)$ is true.
  Then

$$\begin{aligned} LHS(k+1) &= f_0 - f_1 + ... - f_{2k-1} + f_{2k} - f_{2k+1} + f_{2k+2} \\ &= f_{2k-1} - 1 - f_{2k+1} + f_{2k+2} \\ &= f_{2k-1} - 1 - f_{2k+1} + (f_{2k} + f_{2k+1}) \\ &= f_{2k-1} + f_{2k} - 1 \\ &= f_{2k+1} - 1 \end{aligned}$$

  and

$$RHS(k+1) = f_{2k+1} - 1$$

  Therefore $LHS(k+1) = RHS(k+1)$, which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all n.

Extra Credit

Show that $\forall n \in \mathbb{N}, n > 1$, a set S_n with n elements has $\dfrac{n(n-1)(n-2)}{6}$ subsets that contain exactly three elements.

Let $P(n)$ be the proposition: A set S_n with n elements has $\dfrac{n(n-1)(n-2)}{6}$ subsets that contain exactly three elements.
We want to show that $P(n)$ is true for all $n \geq 3$; we use a proof by induction.

- *Basis step:* $P(3)$ is true: In a set S_3 of 3 elements, there is only one subset that containing exactly three elements, and $(3(3-1)(3-2))/6 = 1$.

- *Inductive step:* Let k be a positive integer greater or equal to 3 ($k \geq 3$), and let us suppose that $P(k)$ is true. We want to show that $P(k+1)$ is true.
  Let $S_{k+1} = \{a_1, a_2, \ldots, a_{k+1}\}$ be a set of $k+1$ elements, and let S_k be its subset $S_k = \{a_1, a_2, \ldots, a_k\}$.
  S_k contains k elements: since $P(k)$ is true, it contains $k(k-1)(k-2)/6$ three-element subsets. In addition, based on exercise 7, it also contains $k(k-1)/2$ two-element subsets.
  The subsets of S_{k+1} that contain 3 elements are the subsets of 3 elements of S_k, plus the subsets of 3 elements containing a_{k+1}.
  a_{k+1} can pair with each of the two-element subsets of S_k in order to form a subset of exact three elements of S_{k+1}. These new subsets do not duplicate with any of the other three-element subsets because $a_(k+1)$ does not appear in any of these subsets. There are no other three-element subsets.
  Therefore, the total number N_3 of three-element subsets of S_{k+1} is:

$$\begin{aligned} N_3 &= \frac{k(k-1)(k-2)}{6} + \frac{k(k-1)}{2} \\ &= \frac{k(k-1)[(k-2)+3]}{6} \\ &= \frac{(k+1)k(k-1)}{6} \\ &= \frac{(k+1)((k+1)-1)((k+1)-2)}{6} \end{aligned}$$

  This validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n \geq 2$.