Homework 5 Solutions

ECS 20 (Fall 17)

Patrice Koehl
koehl@cs.ucdavis.edu

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Exercise 1

a) Show that the following statement is true: "If there exist two integers n and m such that $2n^2 + 2n + 1 = 2m$, then $2n = 3$.

Let P be the statement considered. P is an implication of the form $p \rightarrow q$ with p defined as "n and m are integers such that $2n^2 + 2n + 1 = 2m$" and q defined as "$2n = 3$". We prove that p is false, the proposition P is therefore always true.

The proposition p is: there exists two integer n and m are integers such that $2n^2+2n+1 = 2m$. However, we note that:

a) $2n^2 + 2n + 1 = 2(n^2 + n) + 1$, and, since $n^2 + n$ is an integer, $2n^2 + 2n + 1$ is odd.

b) $2m$ is even, as m is an integer

If p were to be true, we would have an odd number equal to an even number... this is a contradiction, and therefore p is false. Since p is false, $p \rightarrow q$ is true.

b) If x and y are rational numbers such that $x < y$, show that there exists a rational number z with $x < z < y$.

This is an existence proof: we only need to find one example.

Let x and y be two rational numbers, then let $z = \frac{x+y}{2}$ which is also rational. Then
$z - x = \frac{x+y}{2} - x = \frac{y-x}{2} > 0$ as $x < y$.
Similarly,
$y - z = y - \frac{x+y}{2} = \frac{y-x}{2} > 0$ as $x < y$.
Therefore $x < z < y$ and z is rational.

Exercise 2

Let x be a real number. Show that $\lfloor \frac{x}{3} \rfloor + \lfloor \frac{x+1}{3} \rfloor + \lfloor \frac{x+2}{3} \rfloor = \lfloor x \rfloor$.

Let $\lfloor x \rfloor = n$, where n is an integer. By definition of floor, we have:
$n \leq x < n + 1$.

Any integer n can either be of the form $3k$ or $3k + 1$ or $3k + 2$ for some integer k. Thus, we consider three cases:

1) There exists an integer k such that $n = 3k$. We can rewrite the inequality above as:

$$\begin{aligned} & 3k \leq x < 3k+1 \\ \implies \quad & k \leq \frac{x}{3} < k + \frac{1}{3} < k+1 \end{aligned}$$

Therefore

$$\lfloor \frac{x}{3} \rfloor = k. \tag{1}$$

Similarly,

$$\begin{aligned} & 3k+1 \leq x+1 < 3k+2 \\ \implies \quad & k < k + \frac{1}{3} \leq \frac{x+1}{3} < k + \frac{2}{3} < k+1 \end{aligned}$$

Therefore

$$\lfloor \frac{x+1}{3} \rfloor = k \tag{2}$$

And,

$$\begin{aligned} & 3k+2 \leq x+2 < 3k+3 \\ \implies \quad & k < k + \frac{2}{3} \leq \frac{x+2}{3} < k+1 \end{aligned}$$

Therefore

$$\lfloor \frac{x+2}{3} \rfloor = k \tag{3}$$

Combining equations (1) and (2) and (3), we get $\lfloor \frac{x}{3} \rfloor + \lfloor \frac{x+1}{3} \rfloor + \frac{x+2}{3} \rfloor = 3k = n = \lfloor x \rfloor$

2) There exists an integer k such that $n = 3k+1$. We can rewrite the inequality above as:

$$\begin{aligned} & 3k+1 \leq x < 3k+2 \\ \implies \quad & k < k + \frac{1}{3} \leq \frac{x}{3} < k + \frac{2}{3} < k+1 \end{aligned}$$

Therefore

$$\lfloor \frac{x}{3} \rfloor = k. \tag{4}$$

Similarly,

$$\begin{aligned} & 3k+2 \leq x+1 < 3k+3 \\ \implies \quad & k < k + \frac{2}{3} \leq \frac{x+1}{3} < k+1 \end{aligned}$$

Therefore

$$\lfloor \frac{x+1}{3} \rfloor = k \tag{5}$$

Similarly,

$$3k + 3 \le x + 2 < 3k + 4$$
$$\implies \quad k + 1 \le \frac{x+2}{3} < k + \frac{4}{3} < k + 2$$

Therefore

$$\lfloor \frac{x+2}{3} \rfloor = k + 1 \tag{6}$$

Combining equations (4), (5) and (6), we get $\lfloor \frac{x}{3} \rfloor + \lfloor \frac{x+1}{3} \rfloor + \lfloor \frac{x+2}{3} \rfloor = k + k + k + 1 = 3k + 1 = n = \lfloor x \rfloor$

3) There exists an integer k such that $n = 3k + 2$. We can rewrite the inequality above as:

$$3k + 2 \le x < 3k + 3$$
$$\implies \quad k < k + \frac{2}{3} \le \frac{x}{3} < k + 1$$

Therefore

$$\lfloor \frac{x}{3} \rfloor = k. \tag{7}$$

Similarly,

$$3k + 3 \le x + 1 < 3k + 4$$
$$\implies \quad k + 1 \le \frac{x+1}{3} < k + \frac{4}{3} < k + 2$$

Therefore

$$\lfloor \frac{x+1}{3} \rfloor = k + 1 \tag{8}$$

Similarly,

$$3k + 4 \le x + 2 < 3k + 5$$
$$\implies \quad k + 1 < k + \frac{4}{3} \le \frac{x+2}{3} < k + \frac{5}{3} < k + 2$$

Therefore

$$\lfloor \frac{x+2}{3} \rfloor = k + 1 \tag{9}$$

Combining equations (7), (8) and (9), we get $\lfloor \frac{x}{3} \rfloor + \lfloor \frac{x+1}{3} \rfloor + \lfloor \frac{x+2}{3} \rfloor = k + k + 1 + k + 1 = 3k + 2 = n = \lfloor x \rfloor$

Exercise 3

Let x be a real number and N an integer greater or equal to 3.
Show that $\lfloor x \rfloor = \lfloor \frac{x}{N} \rfloor + \lfloor \frac{x+1}{N} \rfloor + \ldots + \lfloor \frac{x+N-1}{N} \rfloor$.

We could use a proof by case that generalizes the solution described for exercise 2, using N case; there is however a faster and maybe more elegant solution.

Let us define:

$f(x) = \lfloor x \rfloor - \lfloor \frac{x}{N} \rfloor - \lfloor \frac{x+1}{N} \rfloor - \ldots - \lfloor \frac{x+N-1}{N} \rfloor$

We show first that $f(x)$ is periodic, with period 1. For this, we need to show that:
$\forall x \in \mathbb{R}, \quad f(x+1) = f(x)$

Let x be a real number. Notice that:

$$\begin{aligned}
f(x+1) &= \lfloor x+1 \rfloor - \lfloor \frac{x+1}{N} \rfloor - \lfloor \frac{x+2}{N} \rfloor - \ldots - \lfloor \frac{x+N-1}{N} \rfloor - \lfloor \frac{x+N}{N} \rfloor \\
&= \lfloor x \rfloor + 1 - \lfloor \frac{x+1}{N} \rfloor - \lfloor \frac{x+2}{N} \rfloor - \ldots - \lfloor \frac{x+N-1}{N} \rfloor - \lfloor \frac{x}{N} + 1 \rfloor \\
&= \lfloor x \rfloor + 1 - \lfloor \frac{x+1}{N} \rfloor - \lfloor \frac{x+2}{N} \rfloor - \ldots - \lfloor \frac{x+N-1}{N} \rfloor - \lfloor \frac{x}{N} \rfloor - 1 \\
&= f(x)
\end{aligned}$$

Since this is true with no conditions on x, it is true for all x, and therefore f is periodic, with 1 being one period.

A periodic function needs to be defined only on one period, here in the interval $[0, 1)$. Let x be in this interval. Then:

$$\begin{aligned}
&0 \leq x < 1 \\
&0 \leq \frac{x}{N} < \frac{1}{N} < 1 \\
&0 \leq \frac{x+1}{N} < \frac{1+1}{N} = \frac{2}{N} < 1 \\
&\ldots \\
&0 \leq \frac{x+N-1}{N} < \frac{1+N-1}{N} = \frac{N}{N} = 1
\end{aligned}$$

Therefore $f(x) = 0$.

Since $f(x) = 0$ on one of its period, we have $f(x) = 0 \quad \forall x \in \mathbb{R}$. Therefore:

$\lfloor x \rfloor = \lfloor \frac{x}{N} \rfloor + \lfloor \frac{x+1}{N} \rfloor + \ldots + \lfloor \frac{x+N-1}{N} \rfloor$

Exercise 4

Let x be a real number. Then show that $(\lceil x \rceil - x)(x - \lfloor x \rfloor) \leq \frac{1}{4}$
When x is integer, then $x = \lceil x \rceil = \lfloor x \rfloor$ implies $(\lceil x \rceil - x)(x - \lfloor x \rfloor) = 0 \leq \frac{1}{4}$. If x is not an integer, there exista a real number ϵ such that $x = \lfloor x \rfloor + \epsilon$ where $1 > \epsilon > 0$. Then$(x - \lfloor x \rfloor) = \epsilon$ and

$(\lceil x \rceil - x) = 1 - \epsilon$. Then

$$\begin{aligned}
(\lceil x \rceil - x)\,(x - \lfloor x \rfloor) \quad &= \epsilon(1-\epsilon) \\
&= \epsilon - \epsilon^2 \\
&= \frac{1}{4} - (\epsilon^2 - \epsilon + \frac{1}{4}) \\
&= \frac{1}{4} - (\epsilon - \frac{1}{2})^2 \\
&\leq \frac{1}{4}
\end{aligned}$$

Exercise 5

Let x be a real number. Solve the following equations:

a) $\lfloor x^2 + x - 5 \rfloor = \frac{1}{2}x$

Let x be a real number. We notice first that $\lfloor x^2 + x - 5 \rfloor$ is an integer. Therefore, if x is a solution of the equation then $\frac{1}{2}x$ should also be an integer, let say k. If $x = 2k$, for some integer k, solves the equation, then $(x^2 + x - 5)$ is an integer so $\lfloor x^2 + x - 5 \rfloor = (x^2 + x - 5)$ and $x^2 + x - 5 = k$. This implies

$$\begin{aligned}
& 4k^2 + 2k - 5 = k \\
\Longrightarrow \quad & 4k^2 + k - 5 = 0 \\
\Longrightarrow \quad & 4k^2 - 4k + 5k - 5 = 0 \\
\Longrightarrow \quad & 4k(k-1) + 5(k-1) = 0 \\
\Longrightarrow \quad & (k-1)(4k+5) = 0 \\
\Longrightarrow \quad & k = 1, k = -\frac{5}{4}
\end{aligned}$$

As k is an integer the only solution is $k = 1$ i.e. $x = 2$.

b) $2\lfloor 4 - x \rfloor = 2x + 1$ for $x \in \mathbb{R}$

Let x be a real number that solves the equation. We notice first that $\lfloor 4 - x \rfloor$ is an integer, which we write as k. Then, the equation gives $2k = 2x + 1$, where k is the integer defined before, and therefore $x = k - \frac{1}{2}$. Then

$$\begin{aligned}
& \lfloor 4 - x \rfloor = k \\
\Longrightarrow \quad & \lfloor 4 - (k - \frac{1}{2}) \rfloor = k \\
\Longrightarrow \quad & \lfloor 4 - k + \frac{1}{2} \rfloor = k \\
\Longrightarrow \quad & 4 - k = k \\
\Longrightarrow \quad & k = 2 \\
\Longrightarrow \quad & x = k - \frac{1}{2} = \frac{3}{2}
\end{aligned}$$

So, $x = \frac{3}{2}$ solves the equation.

Extra Credit

Let x and y be two real numbers such that $0 < x \leq y$. We define:

a) The customized arithmetic mean m of x and y: $m = \dfrac{x+2y}{3}$

b) The customized geometric mean g of x and y: $g = x^{\frac{1}{3}} y^{\frac{2}{3}}$

c) The customized harmonic mean h of x and y: $\dfrac{3}{h} = \left(\dfrac{1}{x} + \dfrac{2}{y}\right)$

Show that:

$$x \leq h \leq g \leq m \leq y$$

We will proceed by steps:

a) Let us show first that:

i) $x \leq m \leq y$
Notice that: $m - x = \frac{x+2y-3x}{3} = \frac{2(y-x)}{3} \geq 0$ since $y \geq x$; therefore $m \geq x$.
Similarly, $y - m = \frac{3y-x-2y}{3} = \frac{y-x}{3} \geq 0$; therefore $y \geq m$.

ii) $x \leq g \leq y$
Notice that $g - x = x^{\frac{1}{3}} y^{\frac{2}{3}} - x^{\frac{1}{3}} x^{\frac{2}{3}} = x^{\frac{1}{3}} \left(y^{\frac{2}{3}} - x^{\frac{2}{3}}\right)$. Since $x \leq y$ and $f(x) := x^{\frac{2}{3}}$ is an increasing function of x, $g - x \geq 0$; therefore $g \geq x$.
Similarly, $y - g = y^{\frac{1}{3}} y^{\frac{2}{3}} - x^{\frac{1}{3}} y^{\frac{2}{3}} = y^{\frac{2}{3}} \left(y^{\frac{1}{3}} - x^{\frac{1}{3}}\right)$. Since $x \leq y$ and $f(x) := x^{\frac{1}{3}}$ is an increasing function of x, $y - g \geq 0$; therefore $y \geq g$.

iii) $x \leq h \leq y$
Notice that $\frac{1}{h}$ is the customized arithmetic mean of $\frac{1}{x}$ and $\frac{1}{y}$. From above, we can say that $\frac{1}{y} \leq \frac{1}{h} \leq \frac{1}{x}$ from which we deduce that $x \leq h \leq y$.

b) $g \leq m$
Since, both g and m are positive, therefore
$m - g \geq 0 \iff 27m^3 - 27g^3 \geq 0 \iff (x+2y)^3 - 27xy^2 \geq 0$.
Now, in these kinds of inequalities, it is always helpful to find the special case (possibly by intuition or hit and trial) when the equality holds. Note that when $x = y$, then $m = x = g$ $\implies (x+2y)^3 - 27xy^2 = 0$. So, we can expect that $(x-y)$ is a factor in $(x+2y)^3 - 27xy^2$. Also if the polynomial attains a minimum at $x = y$, then there should be factor $(x-y)^2$ in $(x+2y)^3 - 27xy^2$. Now, we factorize

$$\begin{aligned}
(x+2y)^3 - 27xy^2 \quad &= x^3 + 6x^2y + 12xy^2 + 8y^3 - 27xy^2 \\
&= x^3 + 6x^2y - 15xy^2 + 8y^3 \\
&= x^3 - 2x^2y + xy^2 + 8x^2y - 16xy^2 + 8y^3 \\
&= x(x-y)^2 + 8y(x-y)^2 \\
&= (x+8y)(x-y)^2 \\
&\geq 0
\end{aligned}$$

as $y \geq x > 0$. Hence, $m - g \geq 0$ i.e. $m \geq g$

c) $h \leq g$

We note again that $\frac{1}{h}$ is the customized arithmetic mean of $\frac{1}{x}$ and $\frac{1}{y}$. The customized geometric mean of $\frac{1}{x}$ and $\frac{1}{y}$ is $(\frac{1}{x})^{\frac{1}{3}}(\frac{1}{y})^{\frac{2}{3}} = \frac{1}{x^{\frac{1}{3}}y^{\frac{2}{3}}} = \frac{1}{g}$. From b) above, we have $\frac{1}{g} \leq \frac{1}{h}$, therefore $h \leq g$.

From a), b), and c), we can conclude that $x \leq h \leq g \leq m \leq y$.