Logic
(The language of mathematics)

1) Propositions

__Definition__: A proposition, a statement, is a declarative sentence that is either true (T) or false (F

We refer to T or F as the truth valu of the proposition.
Note that we may not know if a proposition is true or false; we know however that it is either true, or false, but not both.

__Examples__:

Sentence	Is it a proposition?	Truth valu
$1+1=4$	Yes	F
Today is Friday	Yes	it depends...
It will rain tomorrow	Yes	We will kno
$x+1=2$	No (we do not know x)	
I am lying now	No (liar's paradox)	

(see http://en.wikipedia.org/wiki/Liar-parado

Notation:

p: "Today is Friday"

means that p is the proposition "Today is Friday"

2) Compound propositions

Forming new propositions from old ones.

2.1 Negation

Definition: Let p be a proposition. The sentence "It is not the case that p" is another proposition, called the negation of p, denoted $\neg p$, or sometimes $\sim p$. It is read "not p".

Truth table:

p	$\neg p$
T	F
F	T

Examples

Proposition (Truth value)		Negation (Truth value)	
$2+2=4$	(T)	$2+2\neq 4$	(F)
$1=0$	(F)	$1\neq 0$	(T)
All the politicians are crooks		There is at least one politician that is not a crook	

2.2 Conjunction

Definition The conjunction of two proposi[tions] p and q is the compound propos[ition] that we write $p \wedge q$ and read "p and q", which is only true if p is true and q is true.

Truth table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Examples:

Let p: It is below freezing
q: It is snowing

Proposition	sentence
$p \wedge q$	It is below freezing and it is snowin[g]
$\neg p \wedge q$	It is not below freezing but it is snowi[ng]
$p \wedge \neg q$	It is below freezing but it is not snowin[g]

.2 Disjunction

Definition: The disjunction of two propositions p and q is the compound proposition written as $p \vee q$ and read "p or q" that is true if either p is true, q is true, or both are true.

Truth table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Examples:

p: "The butler did it"

q: "The cook dit it"

$p \vee q$: "Either the butler, or the cook did it".

Notes:

$p \wedge \neg p$ is always false

$p \vee \neg p$ is always true

2.4 Exclusive Or

Definition: The exclusive or of two propositions p and q is the compound proposition written $p \oplus q$ that is true if and only if p is true, or q is true, but not when both are true.

Truth table

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Exercise
Write the truth table for the propositions
$p \oplus p$, $p \oplus \neg p$

p	p	$\neg p$	$p \oplus p$	$p \oplus \neg p$
T	T	F	F	T
F	F	T	F	T

5. Logical equivalence. Tautology. Contradiction

Definition: Two propositions p and q are logically equivalent if they always have the same truth values.
We write $p \Leftrightarrow q$

Definition: A compound statement is a tautology if it is always true

Definition A compound statement is a contradiction if it is always false

Examples of equivalence

(a) $\neg(\neg p) \Leftrightarrow p$

(b) $p \vee p \Leftrightarrow p$

(c) $p \wedge p \Leftrightarrow p$

(d) $p \oplus q \Leftrightarrow (p \vee q) \wedge (\neg(p \wedge q))$

Let us prove that (d) is indeed true, using a truth table

p	q	$p \oplus q$	$p \vee q$	$p \wedge q$	$\neg(p \wedge q)$	$(p \vee q) \wedge (\neg(p \wedge$
T	T	F	T	T	F	F
T	F	T	T	F	T	T
F	T	T	T	F	T	T
F	F	F	F	F	T	F

The two propositions always have the same truth values; they are equivalent.

4. Useful properties of compound propositions

De Morgan's laws in logic:

(a) $\neg(p \wedge q) \Longleftrightarrow (\neg p) \vee (\neg q)$

(b) $\neg(p \vee q) \Longleftrightarrow (\neg p) \wedge (\neg q)$

Proofs:

(a)

equivalent

p	q	$p \wedge q$	$\neg(p \wedge q)$	$(\neg p)$	$(\neg q)$	$(\neg p) \vee (\neg$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Proof of (b)

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$(\neg p) \wedge (\neg q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

the two propositions are equivalent!

Example of application

Exercise: Show that $(p \vee q) \vee (\neg p \wedge \neg q)$ is a tautology

Proof:

- First, we show that $p \vee \neg p$ is a tautology

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

$p \vee \neg p$ is always true $\rightarrow$ tautology

- Second, let $A = (p \vee q) \vee (\neg p \wedge \neg q)$

According to Morgan's Law, $\neg p \wedge \neg q \Leftrightarrow \neg(p \vee q)$

Therefore, $A \Leftrightarrow (p \vee q) \vee (\neg(p \vee q))$

let $B = (p \vee q)$

$A \Leftrightarrow B \vee \neg B$

$A \Leftrightarrow T \rightarrow A$ is a tautology

List of important logical equivalences

	Name
$\neg(\neg p) \Leftrightarrow p$	Double negation
$p \vee p \Leftrightarrow p$ $p \wedge p \Leftrightarrow p$	idempotent
$p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$	commutativity
$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$ $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	Associativity
$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	Distributivity
$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	De Morgan's law

$$p \wedge T \Leftrightarrow p$$

$$p \vee T \Leftrightarrow T$$

$$p \wedge F \Leftrightarrow F$$

$$p \vee F \Leftrightarrow p$$

5) Applications

5.1 Boolean searches

AND : match records that contain both terms

OR : match records that contain either term

NOT : match records that do not contain a term

5.2 Logical puzzles

Example : Smullyan island

Logic problems that relate to the inhabitants of the island of knights and knaves created by Smullyan where knights always tell the truth and knaves always lie. Usually you encounter two people, A and B, who address you with one statement each. Based on theses statement, your task is to find what A and B are.

Example

A says : p_A: " The two of us are both knight

B says : p_B: " A is a knave ".

We solve the problem with a table

if A is	if B is	Truth values p_A	Truth values p_B	Is assertion possible?
knight	knight	T	F	No
knight	knave	F	F	No
knave	knight	F	T	Yes
knave	knave	F	T	No

Therefore, A is a knave, and B is a knight.