Discussion 10: Solutions

ECS 20 (Fall 2016)

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Induction

Exercise a

Let $P(n)$ be the proposition:

$$2 - 2 \times 7 + \ldots + 2(-7)^n = \frac{1-(-7)^{n+1}}{4}$$

We want to show that $P(n)$ is true for all $n > 0$. Let us define: $LHS(n) = 2 - 2 \times 7 + \ldots + 2(-7)^n$ and $RHS(n) = \frac{1-(-7)^{n+1}}{4}$.

- *Basic step*:

$$LHS(1) = 2 - 2 \times 7 = -12 \qquad RHS(1) = \frac{1-49}{4} = -12$$

  Therefore $P(1)$ is true.

- *Induction step*: We suppose that $P(k)$ is true, with $1 \leq k$. We want to show that $P(k+1)$ is true.

$$\begin{aligned}
LHS(k+1) &= 2 - 2 \times 7 + \ldots + 2(-7)^k + 2(-7)^{k+1} \\
&= LHS(k) + 2(-7)^{k+1} \\
&= \frac{1-(-7)^{k+1}}{4} + 2(-7)^{k+1} \\
&= \frac{1-(-7)^{k+1} + 8 \times (-7)^{k+1}}{4} \\
&= \frac{1 + 7 \times (-7)^{k+1}}{4} \\
&= \frac{1 - \times(-7)^{k+2}}{4}
\end{aligned}$$

  and

$$RHS(k+1) = \frac{1 - \times(-7)^{k+2}}{4}$$

Therefore $LHS(k+1) = RHS(k+1)$, which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n > 0$.

Exercise b

Let h be a real number with $h > -1$. Let $P(n)$ be the proposition: $1 + nh \leq (1+h)^n$. Let us define $LHS(n) = 1 + nh$ and $RHS(n) = (1+h)^n$. We want to show that $P(n)$ is true for all $n \geq 1$.

- *Basis step:* We show that $P(1)$ is true:

$$LHS(1) = 1 + h$$
$$RHS(1) = 1 + h$$

  Therefore $LHS(1) \leq RHS(1)$ (in fact they are equal) and $P(1)$ is true.

- *Inductive step:* Let k be a positive integer greater or equal to 1 , and let us suppose that $P(k)$ is true. We want to show that $P(k+1)$ is true.

$$LHS(k) = 1 + kh$$

  Since P(k) is true, we know:

$$1 + kh \leq (1+h)^k$$

  Since $h > -1$, $h + 1 > 0$. We can multiply the inequality above by (1+h):
  Therefore

$$\begin{aligned} (1+kh)(1+h) &\leq (1+h)^{k+1} \\ 1 + (k+1)h + kh^2 &\leq RHS(k+1) \end{aligned}$$

  Note that $0 \leq kh^2$, therefore $1 + (k+1)h \leq 1 + (k+1)h + kh^2$, therefore $LHS(k+1) \leq 1 + (k+1)h + kh^2$.
  Combining those two inequalities, we get

$$LHS(k+1) \leq RHS(k+1)$$

  . Therefore $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n \geq 1$.

Exercise c

Let $P(n)$ be the proposition: $n^2 - 1$ is divisible by 8 for n odd, positive.

Since n is odd, positive, there exists a positive integer k such that $n = 2k+1$. Instead of stating that $P(n)$ is true, for n odd, positive, we can rephrase it as $Q(k)$ is true, for $k \geq 0$, where $Q(k)$ is the proposition:
$(2k+1)^2 - 1$ is divisible by 8.

Let us define: $LHS(k) = (2k+1)^2 - 1$.

- *Basic step*:

$$\begin{aligned} LHS(0) &= 0 \\ LHS(1) &= 9 - 1 = 8 \end{aligned}$$

  both are divisible by 8. Therefore $Q(0)$ and $Q(1)$ are true.

- *Induction step*: We suppose that $Q(k)$ is true, with $1 \leq k$. We want to show that $Q(k+1)$ is true.

  Since $Q(k)$ is true, 8 divides $LHS(k)$ and therefore there exists an integer m such that $LHS(k) = 8m$. Let us consider now $LHS(k+1)$:

$$\begin{aligned} LHS(k+1) &= (2k+3)^2 - 1 \\ &= 4k^2 + 12k + 9 - 1 \\ &= 4k^2 + 4k + 1 - 1 + 8k + 8 \\ &= (2k+1) - 1 + 8k + 8 \\ &= LHS(k) + 8k + 8 \\ &= 8m + 8k + 8 \\ &= 8(m + k + 1) \end{aligned}$$

  Therefore 8 divides $LHS(k+1)$, which validates that $Q(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all n positive, odd.

Fibonacci

Let $P(n)$ be the proposition: $f_{n-1}f_{n+1} - f_n^2 = (-1)^n$. We define $f_{n-1}f_{n+1} - f_n^2$ and $RHS(n) = (-1)^n$. We want to show that $P(n)$ is true for all $n > 0$.

- *Basic step*:

$$\begin{aligned} LHS(1) &= f_0 f_2 - f_1^2 = -1 \\ RHS(1) &= -1 \end{aligned}$$

  Therefore $LHS(1) = RHS(1)$ and $P(1)$ is true.

- *Inductive step:* Let k be a positive integer, and let us suppose that $P(k)$ is true. We want to show that $P(k+1)$ is true.

  We know that $P(k)$ is true, i.e. $f_{k-1}f_{k+1} - f_k^2 = (-1)^k$. Let us calculate $LHS(k+1)$.

$$\begin{aligned} LHS(k+1) &= f_k f_{k+2} - f_{k+1}^2 \\ &= f_k(f_k + f_{k+1}) - f_{k+1}^2 \\ &= f_k^2 + f_k f_{k+1} - f_{k+1}^2 \end{aligned}$$

  It seems difficult to bring in $LHS(k)$. But we do have f_k^2 in the equation, and based on $P(k)$ being true, $f_k^2 = f_{k-1}f_{k+1} - (-1)^k$, i.e. $f_k^2 = f_{k-1}f_{k+1} + (-1)^{k+1}$. Replacing in the equation above, we get

$$\begin{aligned} LHS(k+1) &= f_{k-1}f_{k+1} + (-1)^{k+1} + f_k f_{k+1} - f_{k+1}^2 \\ &= (-1)^{k+1} + (f_{k-1} + f_k)f_{k+1} - f_{k+1}^2 \\ &= (-1)^{k+1} + f_{k+1}f_{k+1} - f_{k+1}^2 \\ &= (-1)^{k+1} + f_{k+1}^2 - f_{k+1}^2 \\ &= (-1)^{k+1} \end{aligned}$$

  and

$$RHS(k+1) = (-1)^{k+1}$$

  Therefore $LHS(k+1) = RHS(k+1)$, which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all n.