Discussion 3: Solutions

ECS 20 (Fall 2016)

Patrice Koehl
koehl@cs.ucdavis.edu

October 4, 2016

Exercise 1

Let p and q be two propositions. The proposition p NOR q is true when both p and q are false, and it is false otherwise. It is denoted $p \downarrow q$.

a) Write down the truth table for $p \downarrow q$

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

b) Show that $p \downarrow q$ is equivalent to $\neg(p \vee q)$

p	q	$p \downarrow q$	$p \vee q$	$(\neg(p \vee q)$
T	T	F	T	F
T	F	F	T	F
F	T	F	T	F
F	F	T	F	T

Therefore $p \downarrow q$ is equivalent to $\neg(p \vee q)$

c) Find a compound proposition logically equivalent to $p \rightarrow q$ using only the logical operator $\downarrow$.

p	q	$p \downarrow p$	$(p \downarrow p) \downarrow q$	$((p \downarrow p) \downarrow q) \downarrow ((p \downarrow p) \downarrow q)$	$p \to q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	F	T	T

Exercise 2

Let $P(x)$ be the statement "$x = x^2$". If the domain consists of the integers, what are the truth values of the following statements:

a) $P(0)$

   $P(0)$: $0 = 0^2$: true

b) $P(1)$

   $P(1)$: $1 = 1^2$: true

c) $P(2)$

   $P(2)$: $2 = 2^2$: false

d) $P(-1)$

   $P(-1)$: $-1 = (-1)^2$: false

e) $\exists x\ P(x)$

   The statement is true: $P(1)$ is true: proof by example

f) $\forall x\ P(x)$

   The statement is false: $P(2)$ is false: proof by counter-example

Exercise 3

Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the phrase "It is not the case that.")

a) All dogs have fleas.

   $\forall d \in Dogs, d$ has fleas.

   Negation: There exists a dog that does not have flea.

b) There exists a horse that can add.

   $\exists h \in Horses, h$ can count.

   Negation: All horses cannot add.

c) Every koala can climb.

$\forall k \in Koalas, k$ can climb.

Negation: There exists koala that cannot climb.

d) No monkey can speak French.

$\forall m \in Monkeys, k$ m cannot speak French.

Negation: There is a monkey that can speak French

e) There exists a pig that can swim and catch fish.

$\exists p \in Pigs, p$ can swim and catch fish.

Negation: Every pig either cannot swim, or cannot catch fish.

Exercise 4

a) Let a and b be two integers. Prove that if $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$

We use a proof by contradiction. Let us suppose that $a > \sqrt{n}$ and $b > \sqrt{n}$. Then $ab > n$, i.e. $n > n$. We have reached a contradiction. Therefore the property is true.

b) Prove or disprove that there exists x rational and y irrational such that x^y is irrational.

Let $x = 2$ and $y = \sqrt{2}$. Then $x^y = 2^{\sqrt{2}}$. There are two cases:

- $2^{\sqrt{2}}$ is irrational. We are done
- $2^{\sqrt{2}}$ is rational. Let us define then $x = 2^{\sqrt{2}}$ and $y = \frac{\sqrt{2}}{4}$. Then

$$\begin{aligned} x^y &= (2^{\sqrt{2}})^{\frac{\sqrt{2}}{4}} \\ &= 2^{\frac{\sqrt{2}\sqrt{2}}{4}} \\ &= 2^{\frac{1}{2}} \\ &= \sqrt{2} \end{aligned}$$

  i.e. x^y is irrational.

We have shown that there exists x rational and y irrational such that x^y is irrational but we do not know the values of x and y: non-constructive proof.

c) Show that $\sqrt[3]{2}$ is irrational.

We do a proof by contradiction. Let us suppose that there exists two integers a and b, with $b \neq 0$, such that $\sqrt[3]{2} = \frac{a}{b}$, with $fracab$ reduced, ie. a and b do not have common factors.

Then: $b\sqrt[3]{2} = a$, i.e. $2b^3 = a^3$. Therefore a^3 is even... it is easy to show that then a is even.

Since a is even, there exists an integer k such that $a = 2k$. Therefore $2b^3 = 8k^3$, i.e. $b^3 = 4k^3$, hence b^3 is even, and b is even.

We have reached a contradiction: $\frac{a}{b}$ reduced $\rightarrow$ a is even and b is even. Therefore $\sqrt[3]{2}$ is irrational.

d) There exists no integers a and b such that $21a + 30b = 1$.

We do a proof by contradiction. Let us suppose that there exists two integers a and b such that $21a + 30b = 1$. Then $3(7a + 10b) = 1$. Since $7a + 10b$ is an integer, 1 would be a multiple of 3; we have reached a contradiction. Therefore the property is true.