Discussion 5: Solutions

ECS 20 (Fall 2016)

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Exercise 0: Additional problems on proofs

- **a)** Let x and y be two integers. Show that if $2x + 5y = 14$ and $y \neq 2$, then $x \neq 2$.

  We need to prove an implication of the form $p \rightarrow q$, where p and q are defined as:

  $p : 2x + 5y = 14$ and $y \neq 2$

  $q : x \neq 2$

  We will use a proof by contradiction, namely we will suppose that the property is false, and find that this leads to a contradiction.

  Hypothesis: $p \rightarrow q$ is false, which is equivalent to saying that p is true, AND q is false.

  Therefore, $2x + 5y = 14$ and $y \neq 2$ and $x = 2$. Replacing x by its value in the first equation, we get $4 + 5y = 14$, namely $y = 2$. Therefore we have $y = 2$ and $y \neq 2$: we have reached a contradiction.

  Therefore the hypothesis is false, which means that $p \rightarrow q$ is true.

- **b)** Let x and y be two integers. Show that if $x^2 + y^2$ is odd, then $x + y$ is odd

  We need to prove an implication of the form $p \rightarrow q$, where p and q are defined as:

  $p : x^2 + y^2$ is odd

  $q : x + y$ is odd

  We will use an indirect proof, namely instead of showing that $p \rightarrow q$, we will show the equivalent property $\not q \rightarrow \not p$, where:

  $\not q : x + y$ is even

  $\not p : x^2 + y^2$ is even

  Hypothesis: $\not q$ is true, namely $x + y$ is even. Since $x + y$ is even, $(x + y)^2$ is even (result from class). Therefore there exists an integer k such that $(x + y)^2 = 2k$. We note also that:

  $(x + y)^2 = x^2 + y^2 + 2xy$,

  Therefore,

  $x^2 + y^2 = 2k - 2xy = 2(k - xy)$

  Since $k - xy$ is an integer, we conclude that $x^2 + y^2$ is even, namely that $\not p$ is true.

  We have shown that $\not q \rightarrow \not p$ is true; we can conclude that $p \rightarrow q$ is true.

Exercise 1

To show that f is bijective (or not) from $\mathbb{R}$ to $\mathbb{R}$, we need to check: (i) that it is a function, (ii) that it is one-to-one (injective), and (iii) that it is onto (surjective).

- **a)** $f(x) = 2x + 4$

  (i) f is a function from $\mathbb{R}$ to $\mathbb{R}$, as its domain is $\mathbb{R}$

  (ii) Let us show that f is injective. Let x and y be two real numbers such that $f(x) = f(y)$. Then $2x + 4 = 2y + 4$, therefore $x = y$. f is injective.

  (iii) Let us show that f is surjective. Let y be an element of the co-domain, $\mathbb{R}$. To find if there exists a real number x such that $f(x) = y$, we solve the equation $f(x) = y$, i.e. $2x + 4 = y$. We find $x = \frac{y-4}{2}$, i.e. x exists for each value of y. f is surjective.

  We conclude that f is bijective.

- **b)** $f(x) = x^2 + 1$

  (i) f is a function from $\mathbb{R}$ to $\mathbb{R}$, as its domain is $\mathbb{R}$

  (ii) Is f injective?. Let x and y be two real numbers such that $f(x) = f(y)$. Then $x^2 + 1 = y^2 + 1$, i.e. $x^2 - y^2 = 0$. This leads to $(x - y)(x + y) = 0$, therefore $x = y$ or $x = -y$. For example, $f(1) = f(-1)$: f is not injective; it is therefore not bijective.

- **c)** $f(x) = (x + 1)/(x + 2)$

  (i) f is not a function from $\mathbb{R}$ to $\mathbb{R}$, as it is not defined for $x = -2$. The domain D is $\mathbb{R} - \{-2\}$. It is a function from D to $\mathbb{R}$. Is it a bijection from D to $\mathbb{R}$?

  (ii) Let x and y be two real numbers such that $f(x) = f(y)$. Then $(x + 1)/(x + 2) = (y + 1)/(y + 2)$, i.e. $(x + 1)(y + 2) = (y + 1)(x + 2)$. After development, we get $2x + y = 2y + x$ i.e. $x = y$. The function is injective.

  (iii) Let y be an element of the co-domain, $\mathbb{R}$. To find if there exists a real number x such that $f(x) = y$, we solve the equation $f(x) = y$, i.e. $(x + 1)/(x + 2) = y$. This becomes $x + 1 = y(x + 2)$, i.e. $x(1 - y) = 2y - 1$, which has a solution if and only if $y \neq 1$. Therefore we found one element of the co-domain ($y = 1$) for which we cannot find an element x such that $f(x) = y$. f is not surjective, therefore f is not bijective.

- **d)** $f(x) = (x^2 + 1)/(x^2 + 2)$

  (i) f is a function from $\mathbb{R}$ to $\mathbb{R}$, as its domain is $\mathbb{R}$

  (ii) Is f injective? We note that $f(1) = f(-1)$: f is not injective, therefore f is not bijective.

Exercise 2

Let $S = \{-1, 0, 2, 4, 7\}$. Find $f(S)$ if:

- **a)**. $f(x) = 1$

  Since $f(x) = 1$ for all elements of S, $f(S) = \{1\}$.

- **b)**. $f(x) = 2x + 1$

  $f(-1) = -1$, $f(0) = 1$, $f(2) = 5$, $f(4) = 9$, and $f(7) = 15$. Therefore $f(S) = \{-1, 1, 5, 9, 15\}$.

- **c)**. $f(x) = \lceil \frac{x}{5} \rceil$
  $f(-1) = -1$, $f(0) = 0$, $f(2) = 0$, $f(4) = 0$, and $f(7) = 2$. Therefore $f(S) = \{-1, 0, 1\}$.

- **d)**. $f(x) = \lceil \frac{x^2+1}{3} \rceil$
  $f(-1) = 0$, $f(0) = 0$, $f(2) = 1$, $f(4) = 5$, and $f(7) = 16$. Therefore $f(S) = \{0, 1, 5, 16\}$.

Exercise 3

Let S be a subset of a universe U. The characteristic function f_S of S is the function from U to the set $\{0, 1\}$ such that $f_S(x) = 1$ if x belongs to S and $f_S(x) = 0$ if x does not belong to S. Let A and B be two sets. Show that for all x in U,

- **a)**. $f_{A \bigcap B}(x) = f_A(x) f_B(x)$

  Let x be an element of U. Let us call $LHS(x) = f_{A \bigcap B}(x)$ and $RHS(x) = f_A(x) f_B(x)$. We distinguish two cases:

  (i) $x \in A \bigcap B$. Then $LHS(x) = f_{A \bigcap B}(x) = 1$, by definition of $f_{A \bigcap B}$. Also, since $x \in A \bigcap B$, then $x \in A$ and $x \in B$, therefore $f_A(x) = 1$ and $f_B(x) = 1$, i.e. $RHS(x) = 1$.

  (ii) $x \notin A \bigcap B$. Then $LHS(x) = f_{A \bigcap B}(x) = 0$, by definition of $f_{A \bigcap B}$. Also, since $x \notin A \bigcap B$, then $x \notin A$ or $x \notin B$, therefore $f_A(x) = 0$ or $f_B(x) = 0$, i.e. $RHS(x) = 0$.

  The property is therefore true for all x in U.

- **b)**. $f_{A \bigcup B}(x) = f_A(x) + f_B(x) - f_A(x) f_B(x)$

  Let x be an element of U. Let us call $LHS(x) = f_{A \bigcup B}(x)$ and $RHS(x) = f_A(x) + f_B(x) - f_A(x) f_B(x)$. We distinguish four cases:

  (i) $x \in A$ and $x \in B$. Then $LHS(x) = f_{A \bigcap B}(x) = 1$, as $x \in A \bigcup B$. Also, $f_A(x) = 1$ and $f_B(x) = 1$, therefore $RHS(x) = 1 + 1 - 1 = 1$.

  (ii) $x \in A$ and $x \notin B$. Then $LHS(x) = f_{A \bigcap B}(x) = 1$, as $x \in A \bigcup B$. Also, $f_A(x) = 1$ and $f_B(x) = 0$, therefore $RHS(x) = 1 + 0 - 0 = 1$.

  (iii) $x \notin A$ and $x \in B$. Then $LHS(x) = f_{A \bigcap B}(x) = 1$, as $x \in A \bigcup B$. Also, $f_A(x) = 0$ and $f_B(x) = 1$, therefore $RHS(x) = 0 + 1 - 0 = 1$.

  (iv) $x \notin A$ and $x \notin B$. Then $LHS(x) = f_{A \bigcap B}(x) = 0$, as $x \notin A \bigcup B$. Also, $f_A(x) = 0$ and $f_B(x) = 0$, therefore $RHS(x) = 0 + 0 - 0 = 0$.

  The property is therefore true for all x in U.

Exercise 4

Let n be an integer. Show that $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil = \lfloor \frac{n^2}{4} \rfloor$.

Let n be an integer. We define $LHS(n) = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ and $RHS(n) = \lfloor \frac{n^2}{4} \rfloor$. Since we consider the division of n by 2, we consider two cases:

(i) n **is even**. Then there exists $k \in \mathbb{Z}$ such that $n = 2k$. Then:
$\lfloor \frac{n}{2} \rfloor = k$, $\lceil \frac{n}{2} \rceil = k$, therefore $LHS(n) = k^2$.
$n^2 = 4k^2$, therefore $\lfloor \frac{n^2}{4} \rfloor = k^2$, i.e. $RHS(n) = k^2$.
(ii) n **is odd**. Then there exists $k \in \mathbb{Z}$ such that $n = 2k + 1$. Then:
$\frac{n}{2} = k + \frac{1}{2}$. Then, $\lfloor \frac{n}{2} \rfloor = k$, $\lceil \frac{n}{2} \rceil = k + 1$, therefore $LHS(n) = k^2 + k$.

$n^2 = 4k^2 + 4k + 1$, therefore $\lfloor \frac{n^2}{4} \rfloor = k^2 + k$, i.e. $RHS(n) = k^2 + k$.
The property is therefore true for all n in $\mathbb{Z}$.