Discussion 6: Solutions

ECS 20 (Fall 2016)

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Exercise 1: proofs

$\left\lfloor \frac{\lfloor nx \rfloor}{n} \right\rfloor$

- **a)** Let x and y be two integers. Show that if $2x + 5y = 14$ and $y \neq 2$, then $x \neq 2$.

  We need to prove an implication of the form $p \rightarrow q$, where p and q are defined as:

  $p : 2x + 5y = 14$ and $y \neq 2$

  $q : x \neq 2$

  We will use a proof by contradiction, namely we will suppose that the property is false, and find that this leads to a contradiction.

  Hypothesis: $p \rightarrow q$ is false, which is equivalent to saying that p is true, AND q is false.

  Therefore, $2x + 5y = 14$ and $y \neq 2$ and $x = 2$. Replacing x by its value in the first equation, we get $4 + 5y = 14$, namely $y = 2$. Therefore we have $y = 2$ and $y \neq 2$: we have reached a contradiction.

  Therefore the hypothesis is false, which means that $p \rightarrow q$ is true.

- **b)** Let x and y be two integers. Show that if $x^2 + y^2$ is odd, then $x + y$ is odd

  We need to prove an implication of the form $p \rightarrow q$, where p and q are defined as:

  $p : x^2 + y^2$ is odd

  $q : x + y$ is odd

  We will use an indirect proof, namely instead of showing that $p \rightarrow q$, we will show the equivalent property $\neg q \rightarrow \neg p$, where:

  $\neg q : x + y$ is even

  $\neg p : x^2 + y^2$ is even

  Hypothesis: $\neg q$ is true, namely $x + y$ is even. Since $x + y$ is even, $(x + y)^2$ is even (result from class). Therefore there exists an integer k such that $(x + y)^2 = 2k$. We note also that:

  $(x + y)^2 = x^2 + y^2 + 2xy$,

  Therefore,

$x^2 + y^2 = 2k - 2xy = 2(k - xy)$

Since $k - xy$ is an integer, we conclude that $x^2 + y^2$ is even, namely that $\neg p$ is true.

We have shown that $\neg q \to \neg p$ is true; we can conclude that $p \to q$ is true.

Exercise 2: floor and ceiling

- **a)**. Let x be a real number. Show that:

$$\left\lfloor \frac{\lfloor \frac{x}{2} \rfloor}{2} \right\rfloor = \left\lfloor \frac{x}{4} \right\rfloor$$

Let us define $k = \lfloor \frac{x}{2} \rfloor$ and $m = \lfloor \frac{x}{4} \rfloor$. By definition of floor, we have the two properties:

$k \leq \frac{x}{2} < k + 1$

and

$m \leq \frac{x}{4} < m + 1$

Let us multiply the second inequalities by 2:

$2m \leq \frac{x}{2} < 2(m + 1)$

We notice that:

$k \leq \frac{x}{2}$ and $\frac{x}{2} < 2(m + 1)$; therefore $k < 2(m + 1)$.

$k \leq \frac{x}{2}$ and $2m \leq \frac{x}{2}$. Therefore k and $2m$ are two integers smaller than $\frac{x}{2}$. By definition of floor, k is the largest integer smaller that $\frac{x}{2}$. Therefore $2m \leq k$.

Combining those two inequalities, we get $2m \leq k < 2(m + 1)$. After division by 2, $m < \frac{k}{2} < m + 1$. Therefore m is the floor of $\frac{k}{2}$. Replacing m and k by their values, we get:

$$m = \left\lfloor \frac{x}{4} \right\rfloor = \left\lfloor \frac{k}{2} \right\rfloor = \left\lfloor \frac{\lfloor \frac{x}{2} \rfloor}{2} \right\rfloor$$

The property is therefore true.

- **b)**. Let n be an odd integer. Show that

$$\left\lceil \frac{n^2}{4} \right\rceil = \frac{n^2 + 3}{4}$$

We use a direct proof. As n is an odd integer, there exists an integer k such that $n = 2k + 1$. Then $n^2 = 4k^2 + 4k + 1$. Therefore,

$LHS = \left\lceil \frac{n^2}{4} \right\rceil = \lceil k^2 + k + \frac{1}{4} \rceil = k^2 + k + \lceil \frac{1}{4} \rceil = k^2 + k + 1$

and

$RHS = \frac{n^2+3}{4} = \frac{4k^2+4k+4}{4} = k^2 + k + 1$

Therefore $LHS = RHS$; the property is true.

Exercise 3

- **a)**. Show that if a function $f(x)$ from $\mathbb{R}$ to $\mathbb{R}$ is $O(x)$, then $f(x)$ is $O(x^2)$.

  By definition of O, there exists a real number k and a constant C such that if $x > k$, then $|f(x)| < C|x|$.

  Let $k_2 = \max(k, 1)$. Since $k_2 > k$, we have that for $x > k_2$,

  $|f(x)| < C|x|$

  Since $k_2 > 1$, we have that for $x > k_2$,

  $|x| < |x^2|$

  Combining those two inequalities, we get that for $x > k_2$,

  $|f(x)| < C|x^2|$

  Therefore $f(x)$ is $O(x^2)$.

- **b)**. Show that $f(n) = n\log(n^2+1) + \frac{log(n)}{n^2+1}$ is $O(n\log(n))$.

  Notice first that $f(n)$ can be written as the sum of two functions $g(n) = n\log(n^2+1)$ and $h(n) = \frac{log(n)}{n^2+1}$. Let us work separately with $g(n)$ and $h(n)$:

  i) Notice that:

  $g(n) = n\log(n^2(1+\frac{1}{n^2})) = 2nlog(n) + n\log(1+\frac{1}{n^2})$

  Since $\frac{1}{n^2} < 1$ for $n > 1$, $1 + \frac{1}{n^2} < 2$ and $n\log(1+\frac{1}{n^2}) < n\log(2)$. Therefore $n\log(1+\frac{1}{n^2})$ is $O(n)$. Since $2nlog(n)$ is $O(n\log(n))$, we conclude that $g(n)$ is $O(n\log(n))$.

  ii) Notice that

  $h(n) = \frac{log(n)}{n^2+1} < \frac{n}{n^2+1} < n$

  Therefore $h(n)$ is $O(n)$.

  We found that $g(n)$ is $O(n\log(n))$ and $h(n)$ is $O(n)$: $f(n) = g(n) + h(n)$ is therefore $O(max(n\log(n), n))$, namely $O(n\log(n))$.