ECS20
Final: Review problems

Exercise 1:

Prove the following statements:

(a) If $m^2+n^2 \neq 0$, then $m \neq 0$ or $n \neq 0$
(b) If $m+n$ is odd, then m or n must be even
(c) If mn is even, then m or n must be even

Exercise 2:

Prove that if 6 divides n^3-n, then 6 divides $(n+1)^3-(n+1)$

Exercise 3:

Prove that every prime number n greater than 3 has a square that is of the form $12k+1$. (Hint: use of proof by case, considering the divisibility of n by 6)

Exercise 4:

Prove that the sum of any three consecutive perfect cubes is divisible by 3. (Hint: three consecutive perfect cubes can be written $(n-1)^3$, n^3 and $(n+1)^3$)

Exercise 5:
Simplify:

$$P = (1+a)(1+a^2)(\cdots)(1+a^{2^n})$$

Exercise 6:
Use proof by induction to show that:

a) Let a_k be the sequence defined by $a_k = a_{k-1}+(k+4)$ for $k \geq 2$, where $a_1=5$. Show that

$$a_n = \frac{n(n+9)}{2} \qquad \forall n \geq 1$$

b) $$\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1} \qquad \forall n \geq 1$$

c) Prove that 3 divides $4^n - 1 \qquad \forall n \geq 1$

Exercise 7:
Pigeonhole principle:

a) Given any seven integers, there will be two that have a difference divisible by 6.
b) Given any twelve integers between 0 and 99, there will be two that have a difference equals to a 2-digit integer with repeating digits (such as 11, 22, 33…).

c) Take 6 points in the plane. The points are connected in pairs by edges. The edges are colored in one of two colors. Then we know that there are 3 points such that the edges connecting them have one and the same color.
d) What is the minimum number of students, each of whom comes from one of the 50 states, enrolled in a university to guarantee that there are at least 100 who come from the same state?

Exercise 8:
Counting:

a) How many strings of six lowercase letters from the English alphabet contain:
    a. The letter a?
    b. The letters a and b?
b) How many bit strings contain exactly eight 0s and 10 1s if every 0 must be immediately followed by a 1.
c) How many ways are there to seat six people around a circular table?