# Review for Final

## Exercise 1

(a) We want to prove : If $m^2 + n^2 \neq 0$, then $m \neq 0$ or $n \neq 0$.
Let $P$ and $Q$ be the propositions "$m^2 + n^2 \neq 0$" and "$m \neq 0$ or $n \neq 0$", respectively. To prove $P \to Q$, we use an indirect proof, i.e. we prove that $\neg Q \to \neg P$.
If $\neg Q$ is true, then $m = 0$ and $n = 0$. Then $m^2 + n^2 = 0$, i.e. $\neg P$ is true. Therefore, using an indirect proof, we have shown that if $m^2 + n^2 \neq 0$, then $m \neq 0$ or $n \neq 0$.

(b) We want to prove : If $m + n$ is odd, then $m$ or $n$ must be even.
Let $P$ and $Q$ be the proposition "$m + n$ is odd", and "$m$ or $n$ are even", respectively. To prove $P \to Q$, we use an indirect proof, i.e. we prove that $\neg Q \to \neg P$.
If $\neg Q$ is true, then $m$ and $n$ are odd, i.e. there exist $k$ and $l$ such that $m = 2k + 1$ and $n = 2l + 1$. Then $m + n = 2k + 2l + 2 = 2(k + l + 1)$, i.e. $m + n$ is even, which means $\neg P$ is true. Therefore, using an indirect proof, we have shown that If $m + n$ is odd, then $m$ or $n$ must be even.

(c) We want to prove : If $mn$ is even, then $m$ or $n$ must be even.
Let $P$ and $Q$ be the proposition "$mn$ is even", and "$m$ or $n$ are even", respectively. To prove $P \to Q$, we use an indirect proof, i.e. we prove that $\neg Q \to \neg P$.
If $\neg Q$ is true, then $m$ and $n$ are odd, i.e. there exist $k$ and $l$ such that $m = 2k + 1$ and $n = 2l + 1$. Then $mn = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1$, i.e. $mn$ is odd, which means $\neg P$ is true. Therefore, using an indirect proof, we have shown that If $mn$ is even, then $m$ or $n$ must be even.

## Exercise 2

We want to prove: If $6|(n^3 - n)$, then $6|((n + 1)^3 - (n + 1))$.
Proof : Let $n^3 - n = 6k$, as $6|(n^3 - n)$.

$$\begin{aligned} (n+1)^3 - (n+1) &= n^3 + 3n^2 + 3n + 1 - (n+1) \\ &= n^3 - n + 3n^2 + 3n \\ &= 6k + 3(n^2 + n) \\ &= 6k + 3n(n+1) \end{aligned}$$

Note that either $n$ or $n + 1$ is even, and since the product of an even number by an odd number is even, $n(n + 1)$ is even, i.e. there exists $l$ such that $n(n + 1) = 2l$. Replacing in the equation above, we get:

$$(n+1)^3 - (n+1) = 6k + 6l = 6(k + l)$$

Therefore $6|((n + 1)^3 - (n + 1))$.
Notice that this would have been the inductive step for a proof by induction that for all $n \geq 1$, $6|(n^3 - n)$.

## Exercise 3

Let us assume $p$ is a prime number and $p > 3$. Let us divide $p$ by 6: $p = 6k + n$, where $n \in \{0, 1, 2, 3, 4, 5\}$. We will see that certain values of $n$ are not possible when p is prime:

- $n = 0$: If $n = 0$, $p = 6k$, but p cannot be a multiple of 6, as $p$ is prime.
- $n = 2$ or $n = 4$ : If $n$ is even, then, $n = 2k'$ and $p = 6k + 2k' = 2(3k + k')$, which means that $p$ is even; again, this is not possible as $p$ is prime.
- $n = 3$: If $n = 3$, then $p = 6k + 3 = 3(2k + 1)$, which means that $p$ has a factor, 3, and this is not possible as $p \neq 3$ and $p$ is prime.
- n=1 : $p^2 = (6n+1)^2 = 36n^2 + 12n + 1 = 12(3n^2 + n) + 1 = 12k_1 + 1$, for $k_1 = 3n^2 + n$, where $k_1$ is an integer.
- n=5 : $p^2 = (6n+5)^2 = 36n^2+60n+25 = 12(3n^2+5n+2)+1 = 12k_5+1$, where $k_3 = 3n^2+5n+2$ and $k_3$ is an integer.

Thus, if $p$ is prime and $p > 3$, $p^2$ can be written in the form $12k + 1$, where $k$ is an integer.
Note that we have proved something stronger that what is needed. We have shown that any number that can be written in the form $6n + 1$ or $6n + 5$ has a square that can be written in the form $12k + 1$. Not all these numbers are prime, but all prime numbers are in this form.

## Exercise 4

Let $n$ be an integer, then $n - 1$, $n$ and $n + 1$ are 3 consecutive integers. Using the formula $(a + b)^3 = a^3 + 3a^2b + 3ab^2$, the sum of their cubes can be written as:

$$\begin{aligned}(n-1)^3 + n^3 + (n+1)^3 &= n^3 - 3n^2 + 3n - 1 + n^3 + n^3 + 3n^2 + 3n + 1 \\ &= 3n^3 + 6n \\ &= 3(n^3 + 2n)\end{aligned}$$

Thus the sum of three consecutive perfect cubes can be written as a multiple of 3.

## Exercise 5

Let $P_n$ be defined as $P_n = (1 + a)(1 + a^2) \dots (1 + a^{2^n})$
Let us consider $Q_n = (1 - a)P_n = (1 - a)(1 + a)(1 + a^2) \dots (1 + a^{2^n})$.
We can write: $Q_0 = (1 - a)(1 + a)$, $Q_1 = Q_0(1 + a^2)$, ..., $Q_n = Q_{n-1}(1 + a^{2^n})$.
We evaluate $Q$ step by step:

$$\begin{aligned}Q_0 &= (1-a)(1+a) = (1-a^2) \\ Q_1 &= (1-a^2)(1+a^2) = (1-a^4) \\ Q_2 &= (1-a^4)(1+a^4) = (1-a^8) \\ \dots & \\ Q_n &= (1-a^{2^n})(1+a^{2^n}) = (1-a^{2^{n+1}})\end{aligned}$$

(Note that to be completely rigorous, we would have to justify that this is true for all n by induction).
To get back to $P_n$, we consider two cases:

- $a \neq 1$. Then $P_n = \dfrac{Q_n}{1-a} = \dfrac{1 - a^{2^{n+1}}}{1-a}$.
- $a = 1$. We cannot use $Q$, and we go back to the definition of $P_n$: $P_n = (1+1)(1+1) \dots (1+1) = 2^{n+1}$.

## Exercise 6

a) Let $a_k$ be the sequence defined as $a_k = a_{k-1} + k + 4$, for $k > 1$, with $a_1 = 5$.

Let $P_n$ be the proposition that $a_n = \frac{n(n+9)}{2}$. We want to prove that $P_n$ is true, $\forall n > 0$.

To simplify the notations, we write $\text{LHS}(n) = a_n$ and $\text{RHS}(n) = \frac{n(n+9)}{2}$. $P_n$ is true means $\text{LHS}(n) = \text{RHS}(n)$

- *Basis step:* $n = 1$, then
  $\text{LHS}(1) = a_1 = 5$ and
  $\text{RHS}(1) = \frac{1 * (1+9)}{2} = \frac{10}{2} = 5.$
  hence $P(1)$ is true.
- *Inductive step:* Let us assume that $P_k$ is true for any integer $k > 0$. We want to prove that $P_{k+1}$ is true:

$$\begin{aligned} \text{LHS}(k+1) &= a_{k+1} \\ &= a_k + (k+1) + 4 \\ &= \frac{k(k+9)}{2} + k + 5 \\ &= \frac{k(k+9) + 2k + 10}{2} \\ &= \frac{k^2 + 11k + 10}{2} \\ &= \frac{(k+1)(k+10)}{2} \end{aligned}$$

  and

$$\begin{aligned} \text{RHS}(k+1) &= \frac{(k+1)(k+1+9)}{2} \\ &= \frac{(k+1)(k+10)}{2} \end{aligned}$$

  Therefore $\text{LHS}(k+1) = \text{RHS}(k+1)$, i.e. $P_{k+1}$ is true.

According to the principle of mathematical induction, we can conclude that for all $n \geq 1$, we get $a_n = \frac{n(n+9)}{2}$.

b) Let $P_n$ be the proposition $\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$. We want to show that $P_n$ is true $\forall n > 0$.

To simplify the notations, we write $\text{LHS}(n) = \sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)}$ and $\text{RHS}(n) = \frac{n}{2n+1}$.

$P_n$ is true means $\text{LHS}(n) = \text{RHS}(n)$

- *Basis step:* For $n = 1$,

$$\text{LHS}(1) = \frac{1}{(2-1)(2+1)} = \frac{1}{3}$$

and

$$\text{RHS}(1) = \frac{1}{2 * 1 + 1} = \frac{1}{3}$$

Since LHS(1) = RHS(1), $P_1$ is true.

- *Inductive step:* Let us assume that $P_k$ is true for any integer $k > 0$.

$$\begin{aligned}
\text{LHS}(k+1) &= \sum_{i=1}^{k+1} \frac{1}{(2i-1)(2i+1)} \\
&= \sum_{i=1}^{k} \frac{1}{(2i-1)(2i+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)} \\
&= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \\
&= \frac{k(2k+3)+1}{(2k+1)(2k+3)} \\
&= \frac{2k^2+3k+1}{(2k+1)(2k+3)} \\
&= \frac{(k+1)(2k+1)}{(2k+1)(2k+3)} \\
&= \frac{k+1}{2k+3}
\end{aligned}$$

and

$$\text{RHS}(k+1) = \frac{k+1}{2(k+1)+1} = \frac{k+1}{2k+3}$$

Hence, LHS$(k+1)$ = RHS$(k+1)$, i.e. we have proved that $P_{k+1}$ is true, given $P_k$ is true.

Thus, we have proved by induction that $\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$, $\forall n > 0$.

c) Let $P_n$ be the proposition $3|(4^n - 1)$; we want to show that $P_n$ is true $\forall n > 0$.

- *Basis step*: For $n = 1$, $4^n - 1 = 4 - 1 = 3$ and $3|3$. Hence $P_1$ is true.
- *Inductive step:* Let us assume that $P_k$ is true for any integer $k > 0$.
  Since $3|(4^k - 1)$, there exits $m$ such that $4^k - 1 = 3m$. Then, we can write $(4^{k+1} - 1)$ as, $4 * 4^k - 1 = 4 * (3m + 1) - 1 = 12m + 3 = 3(4m + 1)$. Therefore $3|(4^{k+1} - 1)$.
  Hence, we have proved that $P_{k+1}$ is true, given $P_k$ is true.

Thus, we have proved by induction that $3|(4^n - 1)$, $\forall n > 0$.

## Exercise 7

a) For any 2 integers $a, b$, $6|(a-b)$ if $a \equiv b \pmod 6$, which is equivalent to $a \pmod 6 = b \pmod 6$. In plain English, $6|(a-b)$ if the remainder of the division of $a$ by 6 is equal to the remainder

of the division of $b$ by 6.
There are 6 possible values for the remainder of a division by 6 : $\{0, 1, 2, 3, 4, 5\}$. Let each of these values be a "box", and let the seven given integers by "objects". According to the pigeonhole principle, arranging 7 "objects" into 6 "boxes" will lead to one of the "boxes" containing at least two "objects". These two integers will therefore have the same remainder upon division by 6, and consequently their difference will be divisible by 6.

b) The question can be rewritten as given any 12 integers, there will be at least one pair of integers whose difference is divisible by 11. We follow the same reasoning as above. Let the "boxes" be the possible values for the remainder of the division of an integer by 11. There are 11 such "boxes". When the given 12 integers are arranged into these "boxes", one box will contain at least 2 integers. These two integers will have the same remainder upon division by 11, and consequently their difference will be divisible by 11.

c) This problem is similar to Ramsey's problem I solved in class, in which we showed that among 6 people, there is always a group of 3 mutual friends or a group of 3 strangers. I will follow the same type of proof.
Let A be one of the points. There are 5 vertices that start from A, that can be of one of the 2 colors. According to the Pigeonhole Principle, at least 3 of these edges are of the same color, C1. Let B, C and D be the 3 corresponding points. These 3 points are connected together by a set of 3 edges, BC, BD and CD. We have to consider two cases:

- At least one of the three edges BC, BD, CD is of color C1: let BC be that edge. Then all edges between A,B,C are of color C1!
- All three edges BC, BD and CD have color C2: then B,C,D is the set of points we are looking for!

In all cases, we find 3 points that are connected by 3 edges of the same color.

1. d) Let us reformulate this problem by introducing "boxes" and "objects": We have 50 states which are the "boxes", and M students enrolled in university which are the "objects". According to the Pigeonhole Principle, one of the "boxes" will contain $N = \lceil \frac{M}{50} \rceil$ "objects", i.e. there will be N students coming from the same state. We want $N = 100$. Let us write $M = 50k + l$, with $0 \leq l \leq 49$. If $l = 0$, then $\lceil \frac{M}{50} \rceil = k$, in which case $k = 100$, and $M = 5000$. If $l \neq 0$, then $\lceil \frac{M}{50} \rceil = k - 1$, in which case $k = 99$, and hence, $M = 4950 + l$. The minimum value of $M$ is therefore $M = 4951$.

## Exercise 8

a) Let $E$ be the set of all possible (lowercase) six-letter strings. There are $26^6$ such strings, and therefore $|E| = 26^6$.

- Let $SA$ be the set of all six-letter strings that contain a. The complement of $SA$ in $E$, $\overline{SA}$, is the set of all six-letter string that do not contain a. There are $25^6$ such strings, and therefore $|\overline{SA}| = 25^6$. Using the rule of complement, we find that $|SA| = |E| - |\overline{SA}| = 26^6 - 25^6 = 64775151$.
- Let $SAB$ be the set of all six-letter strings that contain "a" and "b". Again, it is easier to work with the complement of $SAB$. $\overline{SAB}$ is the set of six-letter strings that do not contain "a" or do not contain "b". Let $\overline{SA}$ be the set of 6-letter strings that do not

contain "a", and $\overline{SB}$ be the set of 6-letter strings that do not contain "b". Then, $\overline{SAB} = \overline{SA} \cup \overline{SB}$. Using the general sum rule, we find that $|\overline{SAB}| = |\overline{SA}| + |\overline{SB}| - |\overline{SA} \cap \overline{SB}|$.

- $\overline{SA}$ is the set of six-letter strings that do not contain "a". We have seen above that there are $25^6$ such strings: $|\overline{SA}| = 25^6$.
- $\overline{SB}$ is the set of six-letter strings that do not contain "b". It is easy to see that there are $25^6$ such strings: $|\overline{SB}| = 25^6$.
- $\overline{SA} \cap \overline{SB}$ is the set of six-letter strings that contain neither "a" nor "b". There are $24^6$ such strings: $|\overline{SA} \cap \overline{SB}| = 24^6$.

Therefore $|\overline{SAB}| = 25^6 + 25^6 - 24^6$, and $|SAB| = |E| - |\overline{SAB}| = 26^6 - 25^6 - 25^6 + 24^6 = 11737502$.

b) The question can be rewritten as placing 8 "01" blocks and the 2 extra "1"'s in the bit string. We therefore have 10 objects in all, with 8 copies of "01" and 2 copies of "1". There are $C(10, 8) = 45$ different ways to organize these 10 objects.

c) This problem is completely equivalent to the problem of finding the number of ways to arrange 6 children in a circle. There are 6! ways to arrange 6 people in a line, but if we make this line circular, each string will appear 6 times, "rotated" by 60 degrees. Therefore there are $\frac{6!}{6} = 5! = 5 * 4 * 3 * 2 * 1 = 120$ ways of seating 6 people around a round table.